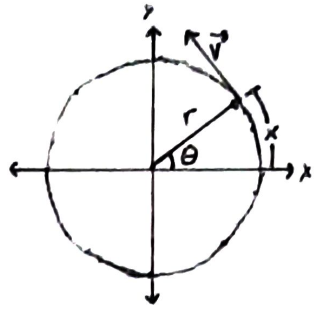
Rotation around an axis

Kinematic equations

Linear		Angular
x	$x = r\theta$	θ
v	$v = r\omega$	$\omega = \frac{d\theta}{dt}$
a	$a = r\alpha$	$\alpha = \frac{d\omega}{dt}$



$$x = r\theta$$

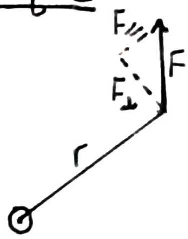
$$\frac{dx}{dt} = r\frac{d\theta}{dt} = v = r\omega$$

$$\frac{dv}{dt} = r\frac{d\omega}{dt} = a = r\alpha$$

If α is constant, you can use kinematic equations.

$$\theta_f = \frac{1}{2}\alpha t^2 + \omega_i t + \theta_i \qquad \omega_f = \alpha t + \omega_i \qquad \omega_f^2 = \omega_i^2 + 2\alpha\Delta\theta$$

Torque



$\perp$ Axis of rotation

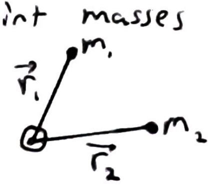
$$\vec{\tau} = \vec{r} \times \vec{F} = rF_{\perp} = rma_{\perp} = rm(r\alpha) = r^2 m\alpha$$

$\boxed{\text{Moment of inertia} \quad I = r^2 m}$

$\boxed{\text{Torque} \quad \vec{\tau} = \vec{r} \times \vec{F} = I\alpha}$

Moment of inertia

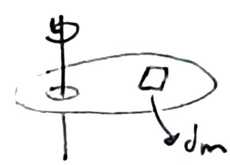
Point masses



$$I = I_1 + I_2 = r_1^2 m_1 + r_2^2 m_2$$

$$\boxed{I = \sum r_i^2 m_i}$$

Continious mass



$$I \approx \sum r_i^2 \Delta m_i$$

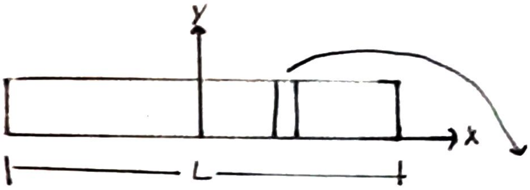
$$I = \lim_{\Delta m \to 0} \sum r_i^2 \Delta m_i$$

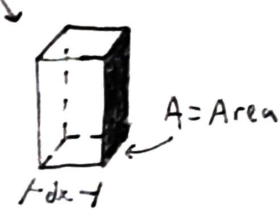
$$\boxed{I = \int r^2 \, dm}$$

Ex: What is I of a rod spinning around its center?

Given M - total mass

L - length





$$dm = \rho dV = \frac{M}{AL} A dx = \frac{M}{L} dx$$

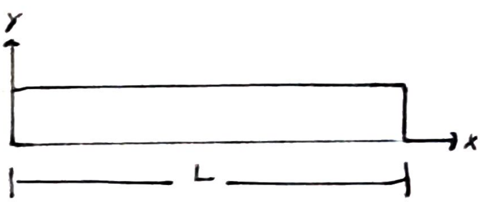
$$dV = A dx$$

$$\rho = \frac{M}{V} = \frac{M}{AL}$$

$$I = \int x^2 dm = \int x^2 \frac{M}{L} dx = \frac{M}{L} \int_{-\frac{L}{2}}^{\frac{L}{2}} x^2 dx = \frac{M}{L}\left[\frac{x^3}{3}\right]_{-\frac{L}{2}}^{\frac{L}{2}} = \frac{M}{L}\left(\frac{\left(\frac{L}{2}\right)^3}{3} - \frac{\left(-\frac{L}{2}\right)^3}{3}\right)$$

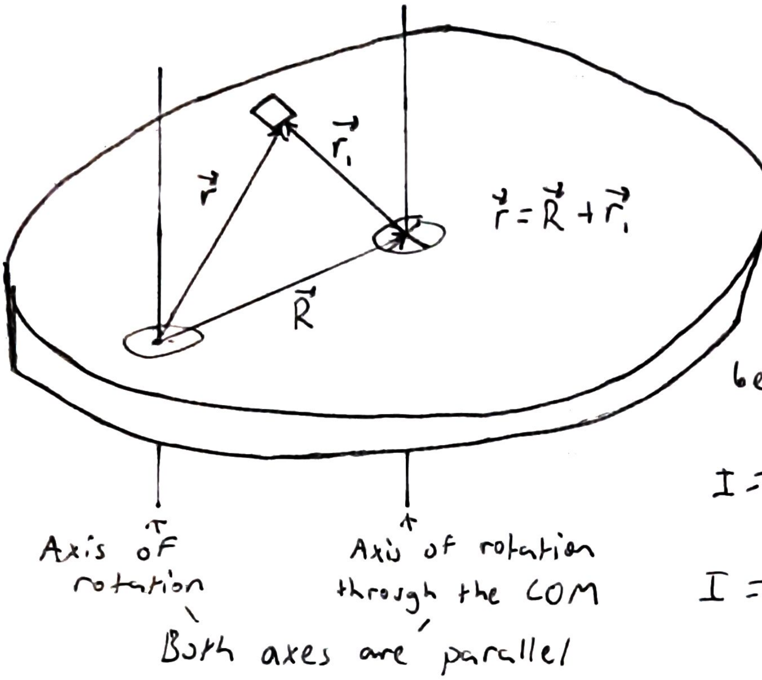
$$= \frac{M}{L}\left(\frac{L^3}{8 \cdot 3} + \frac{L^3}{8\,3}\right) = \frac{M}{L} \cdot \frac{2L^3}{24} = \frac{ML^2}{12}$$

Ex: What is I of a rod spinning around its end?



$$I = \frac{M}{L} \int_{0}^{L} x^2 dx = \frac{M}{L}\left[\frac{x^3}{3}\right]_0^L = \frac{M}{L} \cdot \frac{L^3}{3} = \frac{ML^2}{3}$$

Parallel axis theorem



Given the moment of inertia around the axis of rotation (I) find the moment of inertia around the center of mass (I_{com}). You know the distance between them (R).

$$I = \int r^2 dm = \int \vec{r} \cdot \vec{r}\, dm$$

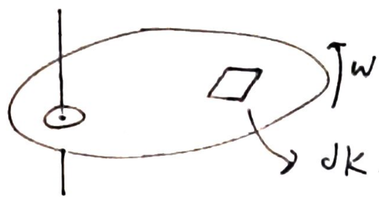
$$I = \int (\vec{R} + \vec{r}_1) \cdot (\vec{R} + \vec{r}_1)\, dm$$

$$I = \int \vec{R} \cdot \vec{R} + \vec{R} \cdot \vec{r}_1 + \vec{r}_1 \cdot \vec{R} + \vec{r}_1 \cdot \vec{r}_1 \, dm = \int R^2 + 2(\vec{R} \cdot \vec{r}_1) + r_1^2 \, dm$$

$$= \int R^2 dm + \int 2(\vec{R} \cdot \vec{r}_1)\, dm + \int r_1^2 dm = R^2 \underbrace{\int dm}_{\text{M - total mass}} + 2\vec{R} \cdot \underbrace{\int \vec{r}_1 dm}_{\text{COM around the COM so } 0} + \underbrace{\int r_1^2 dm}_{I_{COM}}$$

$$\boxed{I = R^2 M + I_{COM}}$$

Rotational kinetic energy



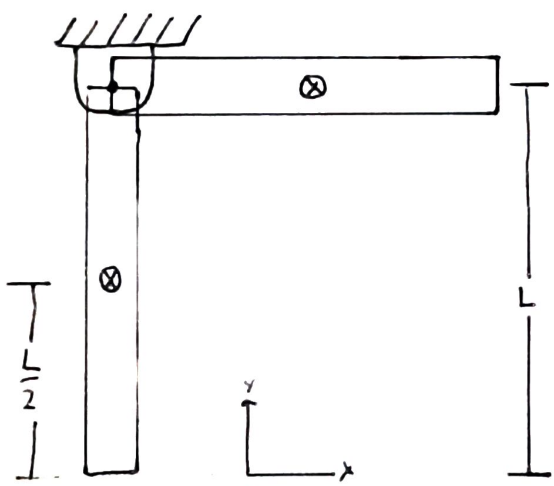
Linear: $K = \frac{1}{2}mv^2$ $\quad v = rw$

$dK = \frac{1}{2}dm\, v^2 = \frac{1}{2}(rw)^2\, dm$

Angular: $K = \int dK = \int \frac{1}{2} r^2 w^2 dm = \frac{1}{2}w^2 \int r^2 dm = \frac{1}{2}w^2 I$

$$\boxed{K = \frac{1}{2}Iw^2}$$

Ex: Given total mass (M) and length (L), what's the final angular velocity at the bottom (w_f)?



$U = 0$ at $y = 0$

$\Delta E = 0 \quad E_i = E_f$

$U_i + K_i = U_f + K_f$

$MgL + 0 = Mg\frac{L}{2} + \frac{1}{2}Iw_f^2$

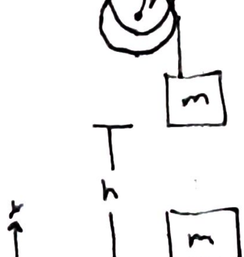
$MgL - MgL\frac{1}{2} = \frac{1}{2}Iw_f^2$

$\frac{1}{2}MgL\left(\frac{2}{I}\right) = w_f^2$

$w_f = \sqrt{\frac{MgL}{I}} \qquad I = \frac{ML^2}{3}$

$w_f = \sqrt{MgL \cdot \frac{3}{ML^2}} = \sqrt{\frac{3g}{L}}$

Ex: What is V_f? Released from rest.



$U = 0$ at $y = 0$

$\Delta E = 0 \quad E_i = E_f$

$U_i + K_i = U_f + K_f$ $\qquad V = rw$

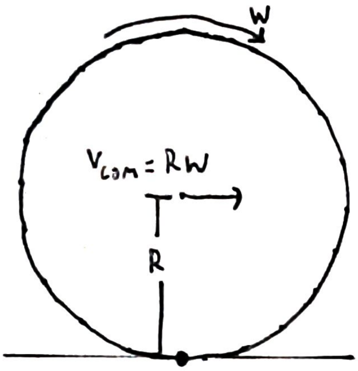
$mgh + 0 = 0 + \frac{1}{2}mV_f^2 + \frac{1}{2}Iw_f^2$ $\qquad w = \frac{V}{r}$

$mgh = \frac{1}{2}mV_f^2 + \frac{1}{2}I\left(\frac{V_f}{r}\right)^2 = \frac{m}{2}V_f^2 + \frac{I}{2r^2}V_f^2$

$V_f = \sqrt{\frac{mgh}{\frac{m}{2} + \frac{I}{2r^2}}}$

Kinetic energy of a rolling wheel

As a wheel rolls, the axis which it rotates around is on the ground.



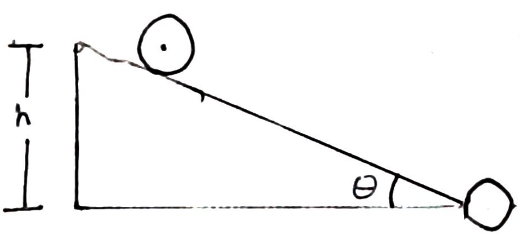
$$K = \frac{1}{2} I W^2 = \frac{1}{2} \underbrace{(MR^2 + I_{com})}_{\text{Parallel axis theorem}} W^2$$

$$= \frac{1}{2} MR^2 W^2 + \frac{1}{2} I_{com} W^2$$

$$= \frac{1}{2} M V_{com}^2 + \frac{1}{2} I_{com} W^2$$

$$\boxed{K_{\text{rolling wheel}} = K_{\text{linear}} + K_{\text{rotational}}}$$

Ex: A wheel of mass M and radius R is released from the top of a ramp. What's its final velocity?



$\Delta E = \emptyset \quad E_i = E_f \quad U_i + K_i = U_f + K_f$

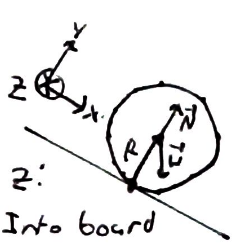
$Mgh + \emptyset = \emptyset + \frac{1}{2} M V_f^2 + \frac{1}{2} I W_f^2$

$V_f = RW_f \quad W_f = \frac{V_f}{R}$

$$Mgh = \frac{1}{2} M V_f^2 + \frac{1}{2} I \frac{V_f^2}{R^2}$$

$$Mgh = \left(\frac{M}{2} + \frac{I}{2R^2}\right) V_f^2 \qquad V_f = \sqrt{\frac{Mgh}{\frac{M}{2} + \frac{I}{2R^2}}}$$

What's the acceleration of the COM?



For z:

$\otimes$ Into board

$\odot$ Out of board

$\vec{\tau}_{net} = I\vec{\alpha} \qquad a = R\alpha \quad \alpha = \frac{a}{R}$

$$\vec{R} \times \vec{W} = RW\sin\theta = I \frac{a}{R} = (MR^2 + I_{com}) \frac{a}{R}$$

$$a = \frac{R^2 Mg \sin\theta}{MR^2 + I_{com}}$$