A massive spring of mass $\underline{M}$, natural length $\underline{L}$, and spring constant $\underline{k}$ is hung vertically from the ceiling. By how much does it stretch under its own weight?

Extra Credit Spring

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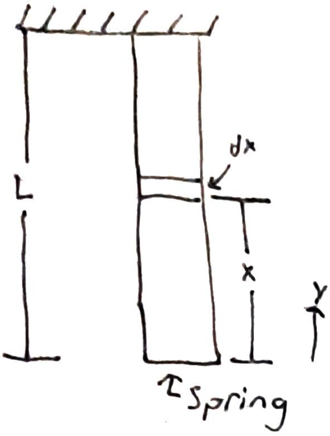

dx = Length of tiny slice of the spring

$d\Delta x$ = Stretch of tiny slice

W = Weight below tiny slice

dk = Spring constant of tiny slice

Step 1) Hooke's law for dx

$$F_{app} = k\,\Delta x \Rightarrow W = dk\, d\Delta x \Rightarrow \boxed{d\Delta x = \frac{W}{dk}}$$

Step 2) Find weight below tiny slice

m = mass below tiny slice

$$W = mg \qquad m = M\frac{x}{L}$$

← The mass below the slice (m) is the spring's total mass (M) multiplied by the fraction of the spring's length that is below the tiny slice

$$= M\frac{x}{L}g$$

$$\boxed{W = \frac{Mgx}{L}}$$

Step 3) Find spring constant of tiny slice

Springs in series: $\frac{1}{K_{total}} = \frac{1}{k_1} + \frac{1}{k_2} + \dots + \frac{1}{k_n}$

$$\frac{1}{K} = \sum_{i=1}^{n} \frac{1}{dk} = n\frac{1}{dk}$$

n = Number of tiny slices

$$n = \frac{L}{dx} \qquad \frac{1}{K} = \frac{L}{dx\, dk} \Rightarrow \boxed{dk = \frac{LK}{dx}}$$

Step 4) Plug in results from steps 2 and 3 into step 1

Step 2: $W = \frac{Mgx}{L}$

step 1: $d\Delta x = \frac{W}{dK}$

Step 3: $dK = \frac{LK}{dx}$

$$d\Delta x = \frac{\frac{Mgx}{L}}{\frac{LK}{dx}} = \frac{Mgx\,dx}{L^2\,K}$$

$$\boxed{d\Delta x = \frac{Mg}{L^2 K}\,x\,dx}$$

step 5) Sum all tiny stetches into the total stetch of the spring

$$\int_0^L d\Delta x = \Delta x \quad = \int_0^L \frac{Mg}{L^2K}\,x\,dx = \frac{Mg}{L^2K}\int_0^L x\,dx = \frac{Mg}{L^2K}\,\frac{L^2}{2}$$

$$\boxed{\Delta x = \frac{Mg}{2K}}$$