4A extra credit problem

After heavy rains, drops of water fall from edge of a roof, at height $\underline{h}$ from the ground, at a rate of $\underline{R}$ per second. Find the center of mass of the drops.

Extra Credit Rain

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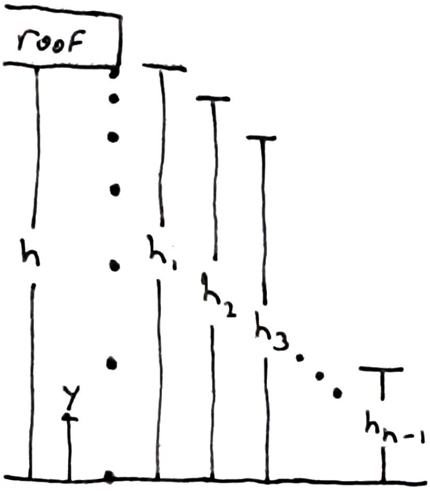


j = Index of drop
h_j = Height of j^{th} drop from the ground
n = Number of drops.

We are assuming that the n^{th} drop is always at the ground. $h_n = \emptyset$

$Y = \emptyset$ at the ground

Step 1) Y center of mass formula in terms of h_j

m = mass of 1 drop
M = mass of all drops
$M = nm$

$$Y_{com} = \frac{1}{M}\sum_{j=1}^{n} h_j m = \frac{1}{mn} m \sum_{j=1}^{n} h_j$$

$$\boxed{Y_{com} = \frac{1}{n}\sum_{j=1}^{n} h_j}$$

Step 2) Use kinematic equations to find h_j in terms of t_j

$Y_f = \frac{1}{2}at^2 + V_i t + Y_i$ $\quad Y_f = h_j \quad a = -g \quad t = t_j \quad V_i = \emptyset \quad Y_i = h$

$h_j = \frac{1}{2}(-g)t_j^2 + \emptyset t_j + h$

$$\boxed{h_j = h - \frac{g}{2}t_j^2}$$

Step 3) Find t_j in terms of R

Δt = Time between drops

If $R = 1$ drops per sec, then the time between drops is 1 sec

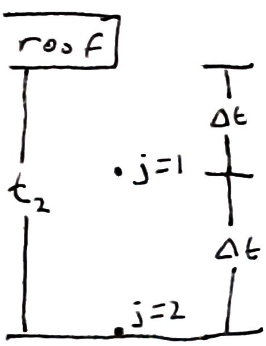
$R=1 \quad \Delta t = 1$

If $R = 2$ drops per sec, then the time between drops is $\frac{1}{2}$ sec

$R=1 \quad \Delta t = \frac{1}{2}$

Therefore, $\quad \Delta t = \frac{1}{R} \quad$ If we assume R is constant, then Δt is also constant.

Let's assume 2 drops have fallen and it's right before another drop falls



$n=2$

Since we're assuming R is constant, we can also assume Δt is constant and therefore

$t_2 = 2\Delta t$ which can be generalized to $t_j = j\Delta t$

substituting $\frac{1}{R}$ for Δt

$$\boxed{t_j = \frac{j}{R}}$$

Step 4) Combine equations from previous steps and simplify

Step 3: $t_j = \frac{j}{R}$

Step 2: $h_j = h - \frac{g}{2} t_j^2 = h - \frac{gj^2}{2R^2}$

Step 1: $Y_{com} = \frac{1}{n} \sum_{j=1}^{n} h_j = \frac{1}{n} \sum_{j=1}^{n} h - \frac{gj^2}{2R^2} = \frac{1}{n} \left(\sum_{j=1}^{n} h - \sum_{j=1}^{n} \frac{g}{2R^2} j^2 \right)$

$= \frac{1}{n} \left(hn - \frac{g}{2R^2} \sum_{j=1}^{n} j^2 \right) = \frac{1}{n} \left(hn - \frac{g}{2R^2} \cdot \frac{n(n+1)(2n+1)}{6} \right)$

$= h - \frac{g(n+1)(2n+1)}{12R^2} = h - \frac{g(2n^2+3n+1)}{12R^2}$

$$\boxed{Y_{com} = h - \frac{2gn^2 + 3gn + g}{12R^2}}$$

Step 5) Find n in terms of R and h

t_n = Time it takes for the n^{th} drop to fall h

From step 3: $t_n = \frac{n}{R} \Rightarrow n = t_n R$

$Y_f = \frac{1}{2} a t^2 + V_i t + Y_i \quad Y_f = 0 \quad a = -g \quad t = t_n \quad V_i = 0 \quad Y_i = h$

$0 = \frac{1}{2}(-g) t_n^2 + 0 t_n + h \Rightarrow -h = -\frac{g}{2} t_n^2 \Rightarrow \frac{2h}{g} = t_n^2 \Rightarrow t_n = \sqrt{\frac{2h}{g}}$

$n = t_n R = R\sqrt{\frac{2h}{g}} = \sqrt{R^2}\sqrt{\frac{2h}{g}} = \sqrt{\frac{2hR^2}{g}}$ $\quad \boxed{n = R\sqrt{\frac{2h}{g}} = \sqrt{\frac{2hR^2}{g}}}$

Step 6) Combine steps 4 and 5 to get Y_{com} in terms of h and R

$$n = \underbrace{R\sqrt{\frac{2h}{g}}}\ = \underbrace{\sqrt{\frac{2hR^2}{g}}}$$

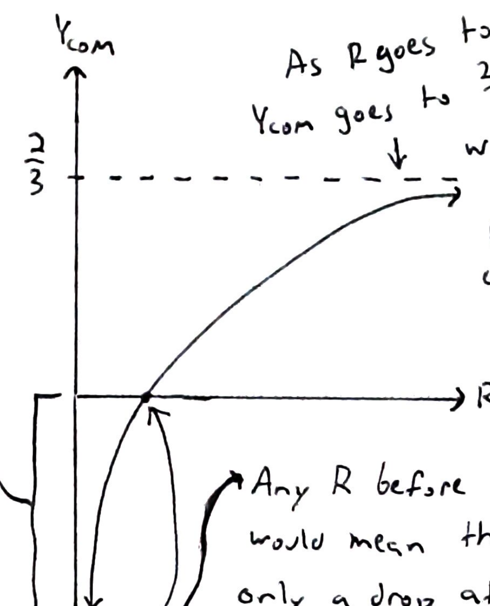
$$Y_{com} = h - \frac{2gn^2 + 3gn + g}{12R^2} = h - \frac{2g\left(\frac{2hR^2}{g}\right) + 3g\left(R\sqrt{\frac{2h}{g}}\right) + g}{12R^2}$$

$$= h - \frac{4hR^2 + 3R\sqrt{g^2}\sqrt{\frac{2h}{g}} + g}{12R^2} = h - \frac{4hR^2}{12R^2} - \frac{3R\sqrt{\frac{2g^2h}{g}}}{12R^2} - \frac{g}{12R^2}$$

$$= h - \frac{h}{3} - \frac{\sqrt{2gh}}{4R} - \frac{g}{12R^2} = \boxed{Y_{com} = \frac{2h}{3} - \frac{\sqrt{2gh}}{4R} - \frac{g}{12R^2}}$$

Step 7) Example with $h = 1m$

$$\underset{\substack{\text{when}\\ h=1}}{Y_{com}} = \frac{2}{3} - \frac{\sqrt{2g}}{4R} - \frac{g}{12R^2}$$



As R goes to ∞, the Y_{com} goes to $\frac{2h}{3}$. This would be the Y_{com} for a continious stream.

But the Y_{com} can't be negative

There is some point $(R, \emptyset)$ where the time between drops is exactly the time a drop takes to fall h.

Any R before that would mean there's only a drop at the bottom so $Y_{com} = \emptyset$

<u>Step 8)</u> Find formula for R at $(R, \emptyset)$ in terms of h.

For any height (h) there is a drop rate (R) such that the time it takes for 1 drop to fall to the ground (t_n) is just before the next drop gets released.

In such a case the Y_{com} would always equal $\emptyset$ because there would only ever be a drop at the bottom.

Note: Our assumption is always $h_n = \emptyset$

What is this R?

We know $t_n = \sqrt{\frac{2h}{g}}$ from step 5.

We know $t_j = \frac{j}{R}$ from step 3.

$t_n = \frac{n}{R}$ $n = 1$ because there's only 1 drop in this example.

$$t_n = \frac{1}{R} = \sqrt{\frac{2h}{g}} \Rightarrow \boxed{R = \sqrt{\frac{g}{2h}}}$$

<u>Step 9)</u> Create the piecewise function

$$\boxed{Y_{com} = \begin{cases} \emptyset & \text{if } R \leq \sqrt{\frac{g}{2h}} \\ \frac{2h}{3} - \frac{\sqrt{2gh}}{4R} - \frac{g}{12R^2} & \text{if } R > \sqrt{\frac{g}{2h}} \end{cases}}$$