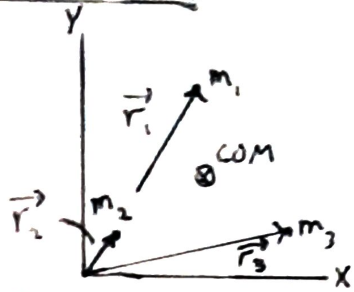
Center of mass

Point masses



$$X_{com} = \frac{X_1 m_1 + X_2 m_2 + X_3 m_3}{m_1 + m_2 + m_3}$$

$$Y_{com} = \frac{Y_1 m_1 + Y_2 m_2 + Y_3 m_3}{m_1 + m_2 + m_3}$$

M: total mass $= m_1 + m_2 + m_3$

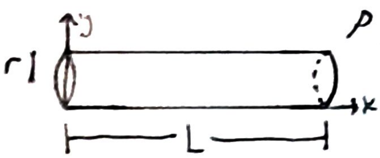
$$\boxed{\vec{r}_{com} = \frac{1}{M} \sum_{i}^{n} \vec{r}_i m_i}$$

Continious masses

$$\boxed{\vec{r}_{com} = \frac{1}{M} \int \vec{r}\, dm} \qquad \boxed{M = \int dm}$$

$$\boxed{\text{Density} = \rho = \frac{m}{V}} \qquad \boxed{\text{Area density} = \sigma = \frac{m}{A}}$$

Ex: What is the COM of a uniform density bar?



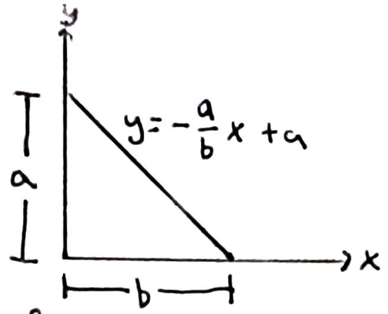
$$X_{com} = \frac{1}{M} \int x\, dm \qquad M = \int dm$$

$dm = \rho\, dV$
$dV = \pi r^2 dx$
$dm = \rho \pi r^2 dx$

$$M = \int_0^L \rho \pi r^2 dx = \rho \pi r^2 \int_0^L dx$$
$$= \rho \pi r^2 L$$

$$X_{com} = \frac{1}{\rho \pi r^2 L} \int_0^L x \rho \pi r^2 dx = \frac{\rho \pi r^2}{\rho \pi r^2 L} \int_0^L x\, dx = \frac{1}{L} \frac{x^2}{2}\Big|_0^L = \frac{L^2}{2L} = \frac{L}{2}$$

Ex: Find the COM of a thin sheet of uniform density.



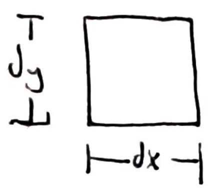
σ is given

$$X_{com} = \frac{1}{M}\int x\,dm$$

$$Y_{com} = \frac{1}{M}\int Y\,dm$$

$M = \int dm$

$$M = \int \sigma\,dy\,dx = \sigma\int dy\,dx$$



$dm = \sigma\,dA$

$dA = dy\,dx$

$$= \sigma\int_0^b \int_0^{-\frac{a}{b}x+a} dy\,dx = \sigma\int_0^b \left(-\frac{a}{b}x + a\right)dx$$

$$= \sigma\left(-\frac{a}{b}\frac{x^2}{2} + ax\right)\Big|_0^b = \sigma\left(-\frac{a}{b}\frac{b^2}{2} + ab\right)$$

$$M = -\frac{1}{2}\sigma ab + \sigma ab = \sigma ab\left(-\frac{1}{2}+1\right) = \frac{\sigma ab}{2}$$

$$\int x\,dm = \int x\,\sigma\,dy\,dx = \sigma\int x\,dx\,dy$$

$y = -\frac{a}{b}x + a \qquad y - a = -\frac{a}{b}x$

$-\frac{b}{a}(y-a) = x \qquad -\frac{b}{a}y + b = x$

$$= \sigma\int_0^a \int_0^{-\frac{b}{a}y+b} x\,dx\,dy = \sigma\int_0^a \frac{x^2}{2}\Big|_0^{-\frac{b}{a}y+b} dy$$

$$= \sigma\int_0^a \frac{\left(-\frac{b}{a}y + b\right)^2}{2}dy = \sigma\int_0^a \frac{\frac{b^2}{a^2}y^2 - \frac{2b^2}{a}y + b^2}{2}dy = \sigma\int_0^a \frac{b^2}{2a^2}y^2 - \frac{b^2}{a}y + \frac{b^2}{2}\,dy$$

$$= \sigma\left(\frac{b^2}{2a^2}\frac{y^3}{3} - \frac{b^2}{a}\frac{y^2}{2} + \frac{b^2}{2}y\right)\Big|_0^a = \sigma\left(\frac{b^2}{2a^2}\frac{a^3}{3} - \frac{b^2}{a}\frac{a^2}{2} + \frac{ab^2}{2}\right)$$

$$= \sigma\left(\frac{ab^2}{6} - \frac{ab^2}{2} + \frac{ab^2}{2}\right) = \frac{\sigma ab^2}{6}$$

$$X_{COM} = \frac{\sigma ab^2}{6}\cdot\frac{2}{\sigma ab} = \frac{1}{3}b \qquad Y_{COM} = \frac{1}{3}a$$