Use cases for integrals:

Net change theorem

$$\int_a^b f'(x)\,dx = f(b) - f(a)$$

The integral of the rate of change ($f'(x)$) is the net change ($f(b) - f(a)$).

Ex: Suppose you have a bird's eye view of a car and know it's x and y velocity in meters per sec.

$V_x(t) = 1$ $\quad$ $V_y(t) = 1 + \cos(2t)$

What is its displacement and distance traveled after 5 sec?

$$\Delta x = \int_0^5 1\,dt = t\Big|_0^5 = 5$$

$$\Delta y = \int_0^5 1 + \cos(2t)\,dt = t + \frac{1}{2}\sin(2t)\Big|_0^5 = 5 + \frac{1}{2}\sin(10)$$

Displacement from origin: $\sqrt{(5)^2 + \left(5 + \frac{1}{2}\sin(10)\right)^2} = 6.88$ meters

Total distance traveled:

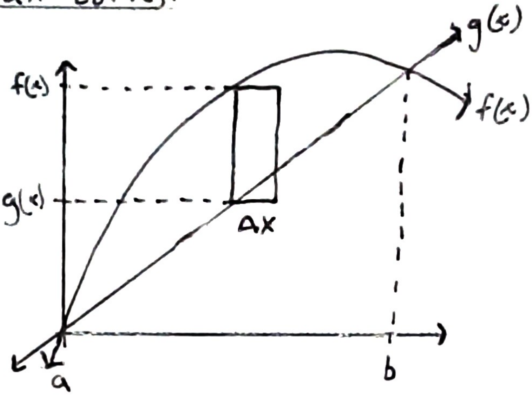
$$\text{Speed}(t) = \sqrt{V_x^2 + V_y^2}$$

$$\int_0^5 \text{Speed}(t)\,dt = \int_0^5 \sqrt{25 + (1 + \cos(2t))^2}\,dt \approx 7.42 \text{ meters}$$

Total distance traveled on the $x = \int_0^5 |V_x|\,dt = 5$ meters

Total distance traveled on the $y = \int_0^5 |V_y|\,dt = 4.73$ meters

Area between curves:



$$\text{Area} = \int_a^b f(x) - g(x)\,dx$$

$$= \left| \int_a^b g(x) - f(x)\,dx \right|$$

1. Find a and b usually by finding intersection points.
2. Find which function is ontop or use absolute value.
3. Solve definite integral

Ex: Find area bounded by $y = \sin x$ and $y = \cos x$ on $[0, \frac{\pi}{2}]$

$\sin x = \cos x \qquad x = \frac{\pi}{4}$

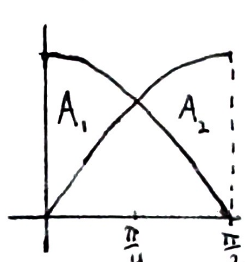
$$\text{Area} = \left| \int_0^{\frac{\pi}{4}} \sin x - \cos x\,dx \right| + \left| \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin x - \cos x\,dx \right|$$

$$= \left| -\cos x - \sin x \Big|_0^{\frac{\pi}{4}} \right| + \left| -\cos x - \sin x \Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}} \right|$$

$$= \left| -\cos\frac{\pi}{4} - \sin\frac{\pi}{4} - (-\cos 0 - \sin 0) \right| + \left| -\cos\frac{\pi}{2} - \sin\frac{\pi}{2} - \left(-\cos\frac{\pi}{4} - \sin\frac{\pi}{4}\right) \right|$$

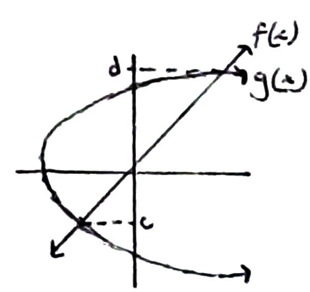
$$= \left| -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} - (-1 - 0) \right| + \left| -0 - 1 - \left(-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\right) \right| = \left| -\sqrt{2} + 1 \right| + \left| -1 + \sqrt{2} \right|$$

$$= \sqrt{2} - 1 + \sqrt{2} - 1 = 2(\sqrt{2} - 1) = 2\sqrt{2} - 2$$



You can also integrate wish respect to y

$$\int_c^d f(x) - g(x)\,dx$$



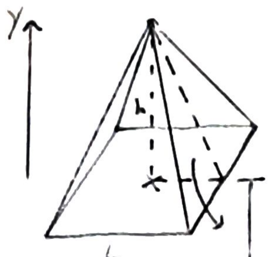
Volumes

Volume of a cylinder = Area of base · Height
(Even for irregular base)

The volume of objects other than cylinders can be found by cutting it into infinitly many cylinders where the area of its base is $A(x)$ or $A(y)$ and its height is dx or dy.

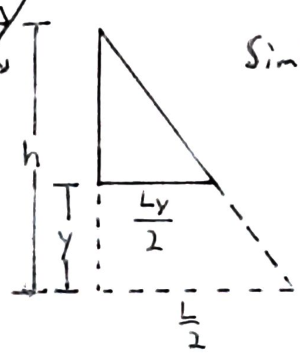
$$\int_a^b A(x)\,dx \quad \text{or} \quad \int_c^d A(y)\,dy$$

Ex 1: Find the volume of a pyramid with a square base of length L and height of h.



$$\int_0^h A(y)\,dy = \int_0^h L_y^2\,dy$$

Cylinders stack in the y direction so it's dy.



Similar triangles:

$$\frac{h}{\frac{L}{2}} = \frac{h-y}{\frac{L_y}{2}} \Rightarrow \frac{2h}{L} = \frac{2(h-y)}{L_y}$$

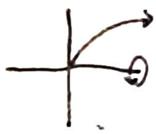
$$L_y = \frac{L}{h}(h-y) = L - \frac{L}{h}y$$

$$\int_0^h \left(L - \frac{L}{h}y\right)^2 dy = \int_0^h L^2 - 2\frac{L^2}{h}y + \frac{L^2}{h^2}y^2\,dy = L^2\int_0^h 1 - \frac{2}{h}y + \frac{1}{h^2}y^2\,dy$$

$$= L^2\left[y - \frac{1}{h}y^2 + \frac{1}{3h^2}y^3\right]_0^h = L^2\left(h - \frac{h^2}{h} + \frac{h^3}{3h^2}\right) = \frac{1}{3}L^2h$$

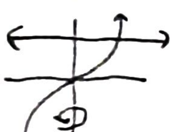
When an area is revolved around an axis, the area of the base is a circle (πr^2) and you need to write r in terms of x or y.

Ex 2: Find the volume of the solid obtained by rotating $y=\sqrt{x}$ around the x-axis from 0 to 1.



$$\int_0^1 \pi(\sqrt{x})^2\,dx = \pi\int_0^1 x\,dx = \pi\left[\frac{1}{2}x^2\right]_0^1 = \frac{\pi}{2}$$

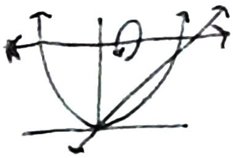
Ex 3: Find the volume of the solid obtained by rotating the area bounded by $y=x^3$, $y=8$, and $x=0$ around the y-axis.



$$\int_0^8 \pi\left(y^{\frac{1}{3}}\right)^2 dy = \pi\int_0^8 y^{\frac{2}{3}}\,dy = \pi\left[\frac{3}{5}y^{\frac{5}{3}}\right]_0^8 = \pi\left(\frac{3}{5}8^{\frac{5}{3}}\right)$$

$$= \pi\frac{3}{5}\left(8^{\frac{1}{3}}\right)^5 = \pi\frac{3}{5}2^5 = \pi\frac{3\cdot 32}{5} = \frac{96}{5}\pi$$

Ex 4: Find the volume of the solid obtained by rotating the area enclosed by $y=x$ and $y=x^2$ around $y=2$.



Find the bounds: $x=x^2 \Rightarrow 0=x^2-x=x(x-1)$

$x=0$ and 1

$$\int_0^1 \pi\underbrace{(2-x^2)}_{\text{Outer radius}}{}^2 - \pi\underbrace{(2-x)}_{\text{Inner radius}}{}^2\,dx = \pi\int_0^1 4-4x^2+x^4-(4-4x+x^2)\,dx$$

$$(2-x^2)(2-x^2) = 4-4x^2+x^4 \qquad (2-x)(2-x) = 4-4x+x^2$$

$$= \pi\int_0^1 x^4-5x^2+4x\,dx = \pi\left[\frac{1}{5}x^5-\frac{5}{3}x^3+\frac{4}{2}x^2\right]_0^1 = \pi\left(\frac{1}{5}-\frac{5}{3}+2\right)$$

$$= \pi\left(\frac{6}{30}-\frac{50}{30}+\frac{60}{30}\right) = \pi\left(-\frac{44}{30}+\frac{60}{30}\right) = \frac{16}{30}\pi = \frac{8}{15}\pi$$