Use cases for derivatives

Find a tangent line at a point

1. Find the slope at the point
   $m = f'(a)$
2. Point slope form: $y - f(a) = m(x-a)$
3. Slope intercept form: $y = mx + b \leftarrow$ y-intercept

Normal line: Perpendicular to the tangent line.
- Slope is the opposite reciprical.

Ex: m of tangent: $\frac{3}{2}$ m of normal: $-\frac{2}{3}$

Related rates: Converting the rate of one variable into another through a common equation.

Ex: Air is being pumped into a spherical balloon so that its volume increases at a rate of 100 cm³/s. How fast is the radius of the balloon increasing when the diameter is 50cm?

$\frac{dV}{dt} = 100 \frac{cm^3}{s}$ $V = \frac{4}{3}\pi r^3$ $\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$

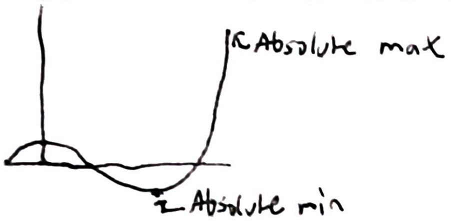
$\frac{dr}{dt} = \frac{dV}{dt} \cdot \frac{1}{4\pi r^2}$ $d = 2r$ $50 = 2r$ $r = 25$

$\frac{dr(r=25)}{dt} = (100) \cdot \frac{1}{4\pi(25)^2} = \frac{1}{2\pi}$ cm/s

Find the absolute max and min values of $f(x)$ in an interval:

0. Graph if possible
1. Check if $f(x)$ is contirious on the interval.
2. Find $f'(x)$
3. Find x values where $f'(x) = 0$ or DNE.
4. Check if those x values are in the interval.
5. Find $f(x)$ for those x values.
6. Find $f(x)$ for the endpoints.
7. Absolute max is the lagest $f(x)$.
   Absolute min is the lowest $f(x)$.

Ex: $f(x) = x^3 - 3x^2 + 1$ on the interval $[-\frac{1}{2}, 4]$.



$f(x)$'s domain is $(-\infty, \infty)$ so it's contirious on the interval.

$f'(x) = 3x^2 - 6x$

$0 = 3x^2 - 6x = (3x)(x-2)$

$x = 0, 2$ which are both in the interval.

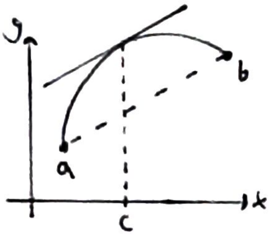
$f(0) = 1$ $f(2) = -3$

Endpoints: $f(-\frac{1}{2}) = \frac{1}{8}$ $f(4) = 17$

Absolute max: $(4, 17)$ Absolute min: $(2, -3)$

Mean value theorem

If f(x) is continious on [a,b] and differentiable on (a,b), then there exists a C in the interval such that f'(c) = the slope of a and b.



$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Ex: Find all Cs that satisfy MVT for $f(x) = x^3 - x$ on (0,2).

f(x) is continious on the interval

$f'(x) = 3x^2 - 1$

$f(0) = 0 \quad f(2) = 6$

$$f'(c) = \frac{f(0) - f(2)}{0 - 2} = \frac{-6}{-2} = 3$$

$3 = 3c^2 - 1 \qquad 4 = 3c^2 \qquad c^2 = \frac{4}{3} \qquad c = \pm\sqrt{\frac{4}{3}}$ so $c = \sqrt{\frac{4}{3}}$

$-\sqrt{\frac{4}{3}}$ isn't in the interval.

Graphing a function

Finding local minimums and maximums

1. Find f'(x)
2. Find x values that cause f'(x) = 0 or DNE
3. Draw number line with x values
4. Pick nums between to see if f'(x) is pos or neg
5. Local min: f'(x) = 0 and f'(x) changes from neg to pos.
   Local max: f'(x) = 0 and f'(x) changes from pos to neg.

Finding inflection points

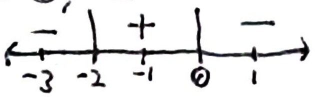
Same steps as local min/max, but with f''(x).

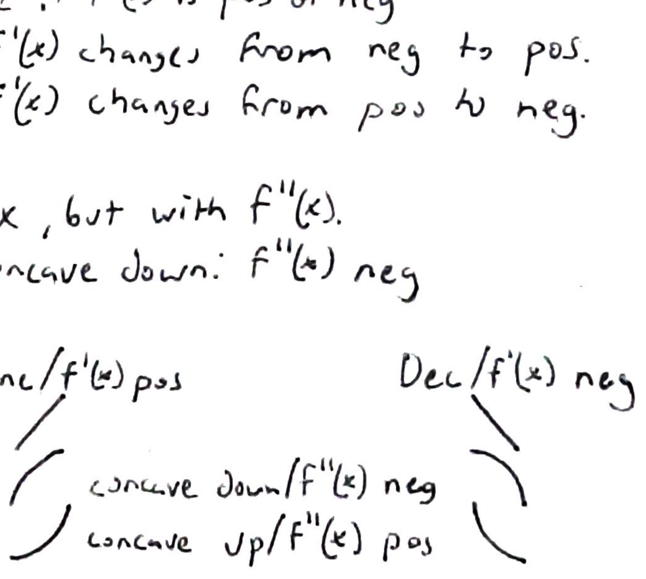
Concave up: f''(x) pos   Concave down: f''(x) neg

Ex: $f(x) = -4x^3 - 12x^2 + 5$

$f'(x) = -12x^2 - 24x = (-12x)(x+2)$

$x = 0, -2$





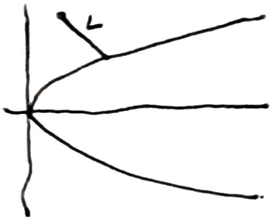
Other useful things for graphing a function:

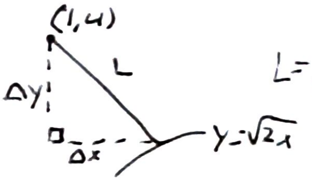
- Finding its domain
- Finding y-intercept (when $x=0$) and x-intercept (when $y=0$)
- Testing symetry
  - If $f(-x)=f(x)$, then it's symetric across the y-axis
  - If $f(-x)=-f(x)$, then it's symetric when rotated 180°
- Finding asymptotes
  - $\lim_{x\to\pm\infty} f(x)=L$ Horizontal asymptote
  - $\lim_{x\to a} f(x)=\pm\infty$ Vertical asymptotes (Test x values that're DNE
- Find local mins/maxes and intervals of inc & dec
- Find points of inflection and concavity

Optimization problems

1. Find equation for var you're optimizing for in terms of one other var.
2. Find local mins/maxes.

Ex: Find the point on $y=\sqrt{2x}$ that's closest to (1,4).





$L=\sqrt{\Delta y^2+\Delta x^2}$ Minimize L

$\Delta y=|4-y|$ $\Delta x=|1-x|$

$\Delta y=|4-\sqrt{2x}|$

$L=\sqrt{(4-\sqrt{2x})^2+(1-x)^2}$

$L=\sqrt{2x-8\sqrt{2x}+16+x^2-2x+1}$

$(4-\sqrt{2x})^2=16-4\sqrt{2x}-4\sqrt{2x}+2x=2x-8\sqrt{2x}+16$

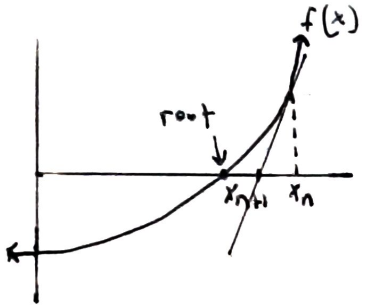
$(1-x)^2=1-x-x+x^2=x^2-2x+1$

$L=\sqrt{x^2-8\sqrt{2x}+17}$

$$L'=\frac{1}{2\sqrt{x^2-8\sqrt{2x}+17}}\left(2x-\frac{8}{2\sqrt{2x}}(2)\right)=\frac{x-\frac{4}{\sqrt{2x}}}{\sqrt{x^2-8\sqrt{2x}+17}}=0 \text{ at } x=2$$

$y=\sqrt{2(2)}=2$ So (2,2) is the point on y closest to (1,4)

Newton's Method:



Used to find roots/zeros of a function.

1. Take a guess of x. $(x_n, f(x_n))$
2. Find tangent line from your guess.

$y - y_1 = m(x - x_1)$

$y - f(x_n) = f'(x_n)(x - x_n)$

3. Solve for x_{n+1} at the point $(x_{n+1}, 0)$

$0 - f(x_n) = f'(x_n)(x_{n+1} - x_n)$

$$\frac{-f(x_n)}{f'(x_n)} = x_{n+1} - x_n$$

$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}}$$

Ex: What is $\sqrt[6]{2}$ to 8 decimal places?

$\sqrt[6]{2} = x$ $\quad 2 = x^6$ $\quad 0 = x^6 - 2 = f(x)$ $\quad f'(x) = 6x^5$

$x_0 = 1.5$

Don't round until the end

$$x_1 = 1.5 - \frac{f(1.5)}{f'(1.5)} = 1.293895748$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 1.170160594$$

$x_3 = 1.127066579$

$x_4 = 1.122508821$

$x_5 = 1.122462053$

$x_6 = 1.122462048$

$x_7 = 1.122462048$

Linear approximation

Uses the tangent line at a point to estimate the function's value nearby that point.

Ex: Find the linearization of $f(x)=\sqrt{x+3}$ at $a=1$ and use it to approximate $\sqrt{3.98}$ and $\sqrt{4.05}$

$f(1)=\sqrt{(1)+3}=\sqrt{4}=2$ $\quad f'(x)=\frac{1}{2\sqrt{x+3}}$ $\quad f'(1)=\frac{1}{2\sqrt{(1)+3}}=\frac{1}{2\sqrt{4}}=\frac{1}{4}$

tangent line: $y-2=\frac{1}{4}(x-1) \Rightarrow y=\frac{1}{4}x+\frac{7}{4}$
/linearization

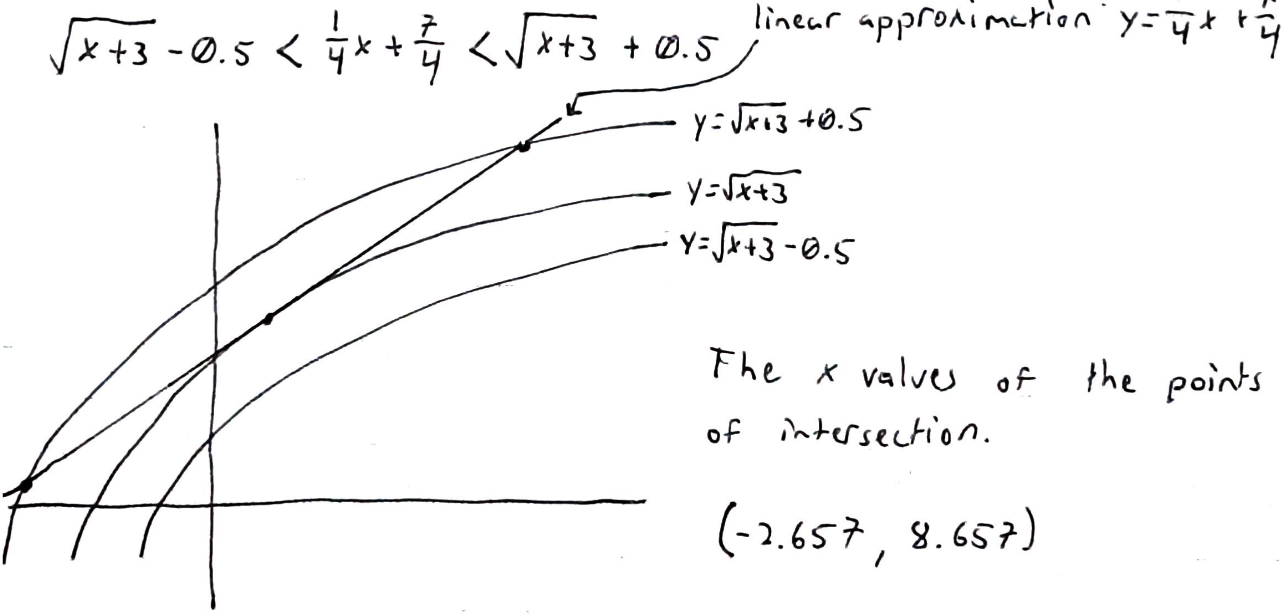
$$\sqrt{3.98}=\sqrt{0.98+3}\approx\frac{1}{4}(0.98)+\frac{7}{4}=1.995$$

$$\sqrt{4.05}=\sqrt{1.05+3}\approx\frac{1}{4}(1.05)+\frac{7}{4}=2.0125$$

For what values of x is the linear approximation accurate to within 0.5?

$$-0.5<\left(\frac{1}{4}x+\frac{7}{4}\right)-\left(\sqrt{x+3}\right)<0.5$$

$$\sqrt{x+3}-0.5<\frac{1}{4}x+\frac{7}{4}<\sqrt{x+3}+0.5$$



The x values of the points of intersection.

$(-2.657, 8.657)$