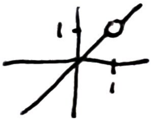
Limit properties:

- If $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$ then $\lim_{x \to a} f(x)$

- The function doesn't need to be defined at the point for the limit to exist



$\lim_{x \to 1} f(x) = 1$

- The limit needs to be a finite number.
  - $\infty / -\infty$ is DNE (Does not exist)

Infinite limits

$\lim_{x \to a} f(x) = \pm\infty$

- Verticle asymptote
- When denominator is $\emptyset$
- $x = a$

Limits at infinity

$\lim_{x \to \pm\infty} f(x) = L$

- Horizontal asymptote
- $y = L$

Finding the limit

1. If the function is continious at the point, then you can use direct substitution.
   - You can do direct substitution for the left and right limits and see if they're equal.

     Ex:
     $$f(x) = \begin{cases} x+1 & \text{if } x \neq 0 \\ 5 & \text{if } x = 0 \end{cases} \qquad \lim_{x \to 0^-} f(x) = 1 \qquad \lim_{x \to 0^+} f(x) = 1$$

   - You can use algebra to convert the function to one that's continious at the point.
     - Factoring: $\lim\limits_{x \to 0} \dfrac{(3+x)^2 - 9}{x} = \lim\limits_{x \to 0} x+6 = 6$

       $$\frac{(3+x)^2 - 9}{x} = \frac{9 + 6x + x^2 - 9}{x} = \frac{x(x+6)}{x} = x+6$$

     - Rationalizing: $\lim\limits_{x \to 0} \dfrac{\sqrt{x^2+9} - 3}{x^2} = \lim\limits_{x \to 0} \dfrac{1}{\sqrt{x^2+9} + 3} = \dfrac{1}{6}$

       $$\frac{\sqrt{x^2+9} - 3}{x^2} \cdot \frac{\sqrt{x^2+9} + 3}{\sqrt{x^2+9} + 3} = \frac{x^2 + 9 - 9}{x^2\left(\sqrt{x^2+9} + 3\right)} = \frac{1}{\sqrt{x^2+9} + 3}$$

     - Least common denominator: $f(x) = \dfrac{1-x}{2+x}$

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\dfrac{1-x-h}{2+x+h} - \dfrac{1-x}{2+x}}{h} \cdot \frac{(2+x+h)(2+x)}{(2+x+h)(2+x)}$$

$$= \lim_{h \to 0} \frac{(1-x-h)(2+x) - (1-x)(2+x+h)}{h(2+x+h)(2+x)}$$

$$(1-x-h)(2+x) = 1(2+x) - x(2+x) - h(2+x) = 2 + x - 2x - x^2 - 2h - xh$$
$$(2+x+h)(1-x) = 2(1-x) + x(1-x) + h(1-x) = 2 - 2x + x - x^2 + h - xh$$

$$\lim_{h \to 0} \frac{\cancel{2} + \cancel{x} - \cancel{2x} - \cancel{x^2} - 2h - \cancel{xh} - (\cancel{2} - \cancel{2x} + \cancel{x} - \cancel{x^2} + h - \cancel{xh})}{h(2+x+h)(2+x)}$$

$$= \lim_{h \to 0} \frac{-2\cancel{h} - \cancel{h}}{\cancel{h}(2+x+h)(2+x)} = -\frac{3}{(2+x)(2+x)} = -\frac{3}{(2+x)^2}$$

2. <u>Squeeze theorem:</u> Finding functions that are greater and less than that converge to the same limit

Ex: $\lim_{x \to 0} x^2 \sin \frac{1}{x}$

$-1 \le \sin x \le 1$

$-1 \le \sin \frac{1}{x} \le 1$

$-x^2 \le x^2 \sin \frac{1}{x} \le x^2$

Since $\lim_{x \to 0} -x^2 = 0$ and $\lim_{x \to 0} x^2 = 0$ and, then $\lim_{x \to 0} x^2 \sin \frac{1}{x} = 0$

3. <u>Composite limit:</u> $\lim_{x \to a} f(g(x)) = f\left(\lim_{x \to a} g(x)\right)$

Ex: $\lim_{x \to 2} (3x+1)^2 = \left(\lim_{x \to 2} (3x+1)\right)^2 = 7^2 = 49$

4. <u>Guess and check</u> from the left and right

- Too far causes calculator errors
- Too short may lead to a wrong guess.

Left	Right
0.9	1.1
0.99	1.01
0.999	1.001
0.9999	1.0001

5. Finding limits at infinity

- Multiplying numerator and denominator by greatest power of x in the denominator.

You want it in the form of $\frac{1}{\infty}$ which $= 0$.

- $\lim_{x \to \infty} |x| = \sqrt{x^2}$ $\qquad \lim_{x \to -\infty} |x| = -\sqrt{x^2}$

- DNE cases
  - Oscilating Ex: $\lim_{x \to \infty} \sin x$
  - $\infty / -\infty$ Ex: $\lim_{x \to \infty} \sqrt{x^2+1}$

Ex: $\lim_{x \to \infty} \sqrt{x^2+1} - x$

$$\frac{\sqrt{x^2+1} - x}{1} \cdot \frac{\sqrt{x^2+1} + x}{\sqrt{x^2+1} + x} = \frac{x^2+1-x^2}{\sqrt{x^2+1} + x}$$

$$\frac{1}{\sqrt{x^2+1} + x} \cdot \frac{\frac{1}{x}}{\frac{1}{x}} = \frac{\frac{1}{x}}{\frac{\sqrt{x^2+1}}{x} + 1} = \frac{\frac{1}{x}}{\frac{\sqrt{x^2+1}}{\sqrt{x^2}} + 1} = \frac{\frac{1}{x}}{\sqrt{\frac{x^2}{x^2} + \frac{1}{x^2}} + 1}$$

$$\lim_{x \to \infty} \frac{\frac{1}{x}}{\sqrt{1 + \frac{1}{x^2}} + 1} = \frac{0}{\sqrt{1} + 1} = 0$$

6. <u>L'Hopital's rule</u>

- When the limit produces $\frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$, then $\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$
  - Simplify the equation after.

Ex: $\lim_{x\to\infty} \frac{\ln x}{\sqrt{x}} \to \frac{\infty}{\infty}$

$$= \lim_{x\to\infty} \frac{\frac{1}{x}}{\frac{1}{2\sqrt{x}}} = \lim_{x\to\infty} \frac{2\sqrt{x}}{x} \to \frac{\infty}{\infty}$$

$$= \lim_{x\to\infty} \frac{\frac{1}{\sqrt{x}}}{1} = \lim_{x\to\infty} \frac{1}{\sqrt{x}} = 0$$

- <u>Indeterminate Products</u>
  - When the limit produces $0 \cdot \pm\infty$, force $f(x)g(x)$ into $\frac{f(x)}{\frac{1}{g(x)}}$ or $\frac{g(x)}{\frac{1}{f(x)}}$

Ex: $\lim_{x\to 0^+} x \ln x \to 0 \cdot -\infty$

Easier taking $\frac{d}{dx}$

$$= \lim_{x\to 0^+} \frac{x}{\frac{1}{\ln x}} \quad \text{or} \quad = \lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x}}$$

$$= \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x\to 0^+} -\frac{x^2}{x} = \lim_{x\to 0^+} -x = 0$$

- <u>Indeterminate Differences</u>
  - When the limit produces $\infty - \infty$, force $f(x) - g(x)$ into a quotient by using common denominators.

Ex: $\lim_{x\to 1^+} \left(\frac{1}{\ln x} - \frac{1}{x-1}\right) \to \infty - \infty$

$$= \lim_{x\to 1^+} \left(\frac{x-1}{\ln x(x-1)} - \frac{\ln x}{\ln x(x-1)}\right) = \lim_{x\to 1^+} \left(\frac{x-1-\ln x}{x\ln x - \ln x}\right) \to \frac{0}{0}$$

$$= \lim_{x\to 1^+} \left(\frac{1-\frac{1}{x}}{\frac{x}{x} + \ln x - \frac{1}{x}}\right) = \lim_{x\to 1^+} \left(\frac{\frac{x}{x} - \frac{1}{x}}{1 + \ln x - \frac{1}{x}}\right) = \lim_{x\to 1^+} \left(\frac{\frac{x-1}{x}}{1 + \ln x - \frac{1}{x}}\right)$$

$$= \lim_{x\to 1^+} \left(\frac{x-1}{x + x\ln x - 1}\right) \to \frac{0}{0} \quad = \lim_{x\to 1^+} \left(\frac{1}{1 + \frac{x}{x} + \ln x}\right) = \frac{1}{2}$$