Finding Integrals

1. Limit definition

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \underbrace{\sum_{i=1}^{n} f(x_i)\,\Delta x}_{\text{Riemann sum}}$$

$f(x_i) = \text{Height}$

$\Delta x = \text{Width} = \dfrac{b-a}{n}$

$x_i = a + i\Delta x$

- We use an infinite number of rectangles to get the area under the curve.
- If we don't use an infinite number, we can use the left, right, midpoint, smallest y value (lower sum), or largest y value (upper sum) to estimate the area under the curve.
- We can also get a rough estimate by finding the max (M) and min (m) of $f(x)$.

If $m \le f(x) \le M$, then $m(b-a) \le \int_a^b f(x)\,dx \le M(b-a)$

$$\sum_{i=1}^{n} c = c\sum_{i=1}^{n} = cn \qquad \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \qquad \sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$$

$$\sum_{i=1}^{n} a_i + b_i = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i \qquad \sum_{i=1}^{n} a_i - b_i = \sum_{i=1}^{n} a_i - \sum_{i=1}^{n} b_i$$

2 Properties of integrals

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx \qquad \int_a^a f(x)\,dx = 0$$

$$\int_a^b c\,f(x)\,dx = c\int_a^b f(x)\,dx \qquad \int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$$

$$\int_a^b \big(f(x)+g(x)\big)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx \qquad \int_a^b \big(f(x)-g(x)\big)\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$$

3. Fundamental theorem of calculus

Part 1: Deriving the integral of f gives f

$$\frac{d}{dx}\int_a^x f(t)\,dt = F(x)$$ a is any constant

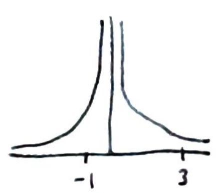
Ex: $\frac{d}{dx}\int_1^{x^4} \sec t \, dt$ $\quad u = x^4 \quad \frac{du}{dx} = 4x^3 \quad \frac{d}{dx} = \frac{du}{dx}\cdot\frac{d}{du}$

$$4x^3 \cdot \frac{d}{du}\int_1^u \sec t \, dt = 4x^3 \cdot \sec u = 4x^3 \sec x^4$$

Part 2:

$$\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - F(a)$$

Ex: $\int_{-1}^{3} \frac{1}{x^2}\,dx \neq -\frac{1}{x}\Big|_{-1}^{3} = -\frac{4}{3}$



f(x) has to be continuous or have a finite number of jump discontinuities over [a, b] to be integrable.

Integration and differentiation are inverse processes.

4. U-Substitution

1. Define u
2. Find $\frac{du}{dx}$
3. Solve for dx
4. Plug u and dx into the integral and simplify

If definite integral: Re-write bounds for u

5. Solve integral

If indefinite integral: Substitute u so it's in terms of x.

Ex 1: $\int x^3 \cos(x^4+2)\,dx$ $\qquad u = x^4+2$

$\frac{du}{dx} = 4x^3 \Rightarrow dx = \frac{du}{4x^3}$

$\int x^3 \cos(u) \frac{du}{4x^3} = \int \frac{1}{4} \cos u \, du = \frac{1}{4}\sin u + C = \frac{1}{4}\sin(x^4+2) + C$

Ex 2: $\int \sqrt{1+x^2}\, x^5 \, dx$ $\qquad u = 1+x^2 \qquad u-1 = x^2$

$\frac{du}{dx} = 2x \Rightarrow dx = \frac{du}{2x}$

$\int \sqrt{u}\, x^5 \frac{du}{2x} = \int \sqrt{u}\, x^4 \frac{1}{2} du = \int \frac{1}{2}\sqrt{u}(u-1)^2 du$

$= \int \frac{1}{2}\sqrt{u}(u^2-2u+1)\,du = \int \frac{1}{2}u^{\frac{5}{2}} - u^{\frac{3}{2}} + \frac{1}{2}u^{\frac{1}{2}}\,du$

$= \frac{1}{2}\cdot\frac{2}{7}u^{\frac{7}{2}} - \frac{2}{5}u^{\frac{5}{2}} + \frac{1}{2}\cdot\frac{2}{3}u^{\frac{3}{2}} + C = \frac{1}{7}(1+x^2)^{\frac{7}{2}} - \frac{2}{5}(1+x^2)^{\frac{5}{2}} + \frac{1}{3}(1+x^2)^{\frac{3}{2}} + C$

Ex 3: $\int \tan x \, dx = \int \frac{\sin x}{\cos x} dx$ $\qquad u = \cos x$

$\frac{du}{dx} = -\sin x \Rightarrow dx = -\frac{du}{\sin x}$

$\int \frac{\sin x}{u}\left(-\frac{du}{\sin x}\right) = \int -\frac{1}{u} du = -\ln|u| + C = -\ln|\cos x| + C$

Ex 4: $\int_1^e \frac{\ln x}{x} dx$ $\qquad u = \ln x$

$\frac{du}{dx} = \frac{1}{x} \Rightarrow dx = x\,du$

Lower: $u = \ln(1) = 0$ $\qquad$ Upper: $u = \ln(e) = 1$

$\int_0^1 \frac{u}{x} x\,du = \int_0^1 u\,du = \frac{1}{2}u^2\Big|_0^1 = \frac{1}{2}(1)^2 - \frac{1}{2}(0)^2 = \frac{1}{2}$