Finding derivatives

1. Use limit definition

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \quad \text{or} \quad \lim_{p \to x} \frac{f(p) - f(x)}{p - x}$$

2. Power rule

$$f(x) = x^n \qquad f'(x) = n x^{n-1}$$

$$x^n \cdot x^m = x^{n+m} \qquad (x^n)^m = x^{nm} \qquad x^{-n} = \frac{1}{x^n}$$

$$\frac{x^n}{x^m} = x^{n-m} \qquad \sqrt[n]{x^m} = x^{\frac{m}{n}} \qquad \frac{1}{x^{-n}} = \frac{1}{\frac{1}{x^n}} = x^n$$

3. $f(x) = e^x \qquad f'(x) = x' e^x$

4. Product rule

$$\frac{d}{dx}\left(f(x) \cdot g(x)\right) = f(x) g'(x) + f'(x) g(x)$$

5. Quotient rule

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x) f'(x) - g'(x) f(x)}{g(x)^2}$$

Denominator

6. Derivative of trig functions

$$\frac{d}{dx} \sin(x) = \cos(x) \qquad \frac{d}{dx} \csc(x) = -\csc(x) \cot(x)$$

$$\frac{d}{dx} \cos(x) = -\sin(x) \qquad \frac{d}{dx} \sec(x) = \sec(x) \tan(x)$$

$$\frac{d}{dx} \tan(x) = \sec^2(x) \qquad \frac{d}{dx} \cot(x) = -\csc^2(x)$$

7. Chain rule

$$\frac{d}{dx} f(g(x)) = f'(g(x))\, g'(x)$$

Ex: $H(x) = \sqrt{x^2+1} = (x^2+1)^{\frac{1}{2}}$

$$H'(x) = \frac{1}{2}(x^2+1)^{-\frac{1}{2}} \frac{d}{dx}(x^2+1)$$

$$= \frac{1}{2}(x^2+1)^{-\frac{1}{2}}(2x)$$

8. $f(x) = b^x$ where $b > 0$ $\quad f'(x) = b^x \ln(b)$

9. Derivative of inverse trig functions

$\frac{d}{dx}\left(\sin^{-1} x\right) = \frac{1}{\sqrt{1-x^2}}$ $\quad \frac{d}{dx}\left(\csc^{-1} x\right) = -\frac{1}{x\sqrt{x^2-1}}$

$\frac{d}{dx}\left(\cos^{-1} x\right) = -\frac{1}{\sqrt{1-x^2}}$ $\quad \frac{d}{dx}\left(\sec^{-1} x\right) = \frac{1}{x\sqrt{x^2-1}}$

$\frac{d}{dx}\left(\tan^{-1} x\right) = \frac{1}{1+x^2}$ $\quad \frac{d}{dx}\left(\cot^{-1} x\right) = -\frac{1}{1+x^2}$

10. Implicit differentiation

- Derive an equation without isolating y

Ex: $x^4 + y^4 = 16$ Find y'

$\frac{d}{dx}\left(x^4 + y^4\right) = \frac{d}{dx}(16)$

$4x^3 + 4y^3 y' = 0$

$4y^3 y' = -4x^3$

$y' = \frac{-4x^3}{4y^4} = -\frac{x^3}{y^4}$

Find y''

$\frac{d}{dx} y' = \frac{d}{dx}\left(-\frac{x^3}{y^3}\right)$

$y'' = -\frac{y^3 \frac{d}{dx} x^3 - x^3 \frac{d}{dx} y^3}{(y^3)^2}$

$= -\frac{y^3 3x^2 - x^3 3y^2 y'}{y^6}$

$= -\frac{y^3 3x^2 - x^3 3y^2 \left(-\frac{x^3}{y^3}\right)}{y^6}$

11. $\frac{d}{dx}\left(\log_b x\right) = \frac{1}{x \ln b}$ $\qquad$ $\frac{d}{dx} \ln x = \frac{1}{x}$

12. Logarithmic differentiation

- Taking ln of both sides to use its properties to simplify taking the derivative.
- Property of logs if a & b > 0

$$\log(ab) = \log(a) + \log(b) \qquad \log(a^n) = n\log(a)$$

$$\log\left(\frac{a}{b}\right) = \log(a) - \log(b) \qquad \log_b(a) = \frac{\ln(a)}{\ln(b)}$$

- $\log_e = \ln$ is used instead of other bases because its derivative is simple.
- Ex:

Differentiate $y = \dfrac{x^{\frac{3}{4}}\sqrt{x^2+1}}{(3x+2)^5}$

$$\ln y = \ln\left(\frac{x^{\frac{3}{4}}\sqrt{x^2+1}}{(3x+2)^5}\right) = \ln\left(x^{\frac{3}{4}}\sqrt{x^2+1}\right) - \ln(3x+2)^5$$

$$= \ln\left(x^{\frac{3}{4}}\right) + \ln\sqrt{x^2+1} - 5\ln(3x+2)$$

$$= \frac{3}{4}\ln x + \frac{1}{2}\ln(x^2+1) - 5\ln(3x+2)$$

$$\frac{d}{dx}\ln y = \frac{d}{dx}\left(\frac{3}{4}\ln x + \frac{1}{2}\ln(x^2+1) - 5\ln(3x+2)\right)$$

$$\frac{1}{y}y' = \frac{3}{4x} + \frac{1}{2(x^2+1)}\frac{d}{dx}(x^2+1) - \frac{5}{3x+2}\frac{d}{dx}(3x+2)$$

$$y' = y\left(\frac{3}{4x} + \frac{x}{x^2+1} - \frac{15}{3x+2}\right)$$

substitute for y if necessary