Differential Equations

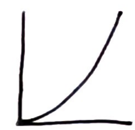
Differential equation - An equation that contains an unknown function and some of its derivatives

Ex: Population growth can be modeled with this differential equation

$\frac{dP}{dt} = kP$ where k is the relative growth rate.

What function whos derivative is a constant multiple of itself?

$P(t) = Ce^{kt}$ $\quad P'(t) = k(Ce^{kt}) = kP$



$P(0) = Ce^{k0} = C$ so C is the starting population

But populations usually level off once they reach their carrying capacity (M)

Logistic differential equation: $\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right)$

$\frac{dP}{dt} \simeq kP$ when P is small $\qquad \frac{dP}{dt} < 0$ when $P > M$

Logistic growth equation: $P(t) = \frac{M}{1 + Ae^{-kt}}$ where $A = \frac{M - P(0)}{P(0)}$

Ex 2: The motion of a spring

Force of spring $= -kx = ma = m\frac{d^2x}{dt^2}$

$\frac{d^2x}{dt^2} = -\frac{k}{m}x$ $\quad$ what function whos 2nd derivative is a constant multiplied by the negative of itself?

Sin or cos

Direction/Slope Fields

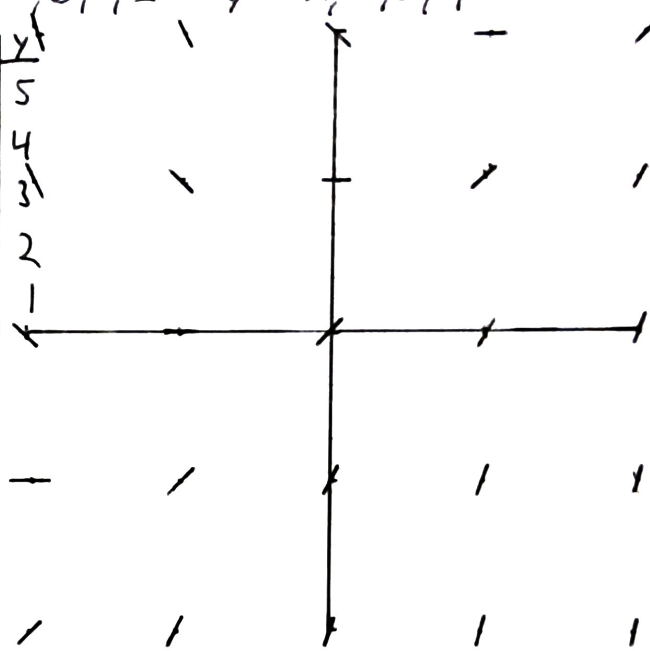
Drawing short line segments with the slope at many points.

Ex: $y' = x - y + 1$ $\quad x = -2, -1, 0, 1, 2 \quad y = -2, -1, 0, 1, 2$

x	y	y'
-2	-2	1
-2	-1	0
-2	0	-1
-2	1	-2
-2	2	-3
-1	-2	2
-1	-1	1
-1	0	0
-1	1	-1
-1	2	-2

x	y	y'
0	-2	3
0	-1	2
0	0	1
0	1	0
0	2	-1
1	-2	4
1	-1	3
1	0	2
1	1	1
1	2	0

x	y	y'
2	-2	5
2	-1	4
2	0	3
2	1	2
2	2	1



Euler's Method

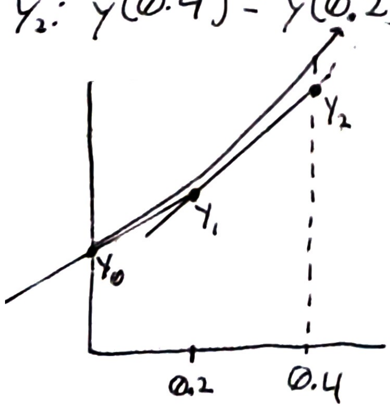
Approximating a value of the function from an initial value.

$$Y_1 = Y_0 + \Delta x\, f'(X_0, Y_0)$$

Ex: $y' = y \quad y(0) = 1 \quad y(0.4) = ? \qquad \Delta x = 0.2$

Y_1: $y(0.2) = y(0) + \Delta x\, y'(0, 1) = 1 + 0.2(1) = 1.2$

Y_2: $y(0.4) = y(0.2) + \Delta x\, y'(0.2, 1.2) = 1.2 + 0.2(1.2) = 1.44$



$y = e^x \qquad e^{0.4} \approx 1.44$

The more steps, the further the error

Separable Equations

1^{st} order differential equations that can be factored into $g(x)f(y)$

$$\frac{dy}{dx} = g(x)f(y) \Rightarrow \frac{1}{f(y)}dy = g(x)dx \Rightarrow \int \frac{1}{f(y)}dy = \int g(x)dx$$

Ex: $\frac{dy}{dx} = \frac{x^2}{y^2} \Rightarrow y^2 dy = x^2 dx \Rightarrow \int y^2 dy = \int x^2 dx$

$$\Rightarrow \frac{y^3}{3} + C_1 = \frac{x^3}{3} + C_2 \Rightarrow \frac{y^3}{3} = \frac{x^3}{3} + (C_2 - C_1) \Rightarrow y^3 = x^3 + 3(C_2 - C_1)$$

$$C = 3(C_2 - C_1) \Rightarrow y^3 = x^3 + C \Rightarrow y = \sqrt[3]{x^3 + C}$$

Ex 2: A tank contains 1000 L of brine with 15 kg of dissolved salt. Pure water enters the tank at a rate of 10 L/min. The solution is kept thoroughly mixed and drains from the tank at the same rate. Find $s(t)$ - How much salt in the tank.

$$\frac{ds}{dt} = (\text{rate in}) - (\text{rate out}) = 0 - \left(\frac{s \text{ kg}}{1000 \text{ L}} \cdot \frac{10 \text{ L}}{1 \text{ min}}\right) = -\frac{s}{100} \frac{\text{kg}}{\text{min}}$$

$s(0) = 15$ kg $\qquad \frac{ds}{dt} = -\frac{s}{100} \Rightarrow \frac{1}{s}ds = -\frac{1}{100}dt$

$$\int \frac{1}{s}ds = \int -\frac{1}{100}dt \Rightarrow \ln|s| = -\frac{t}{100} + C \Rightarrow s = e^{-\frac{t}{100} + C}$$

$$s = e^C e^{-\frac{t}{100}} \qquad A = e^C \Rightarrow s = Ae^{-\frac{t}{100}}$$

$$s(0) = 15 = Ae^0 = A \qquad A = 15$$

$$s(t) = 15e^{-\frac{t}{100}} \text{ kg}$$

Orthogonal Trajectories

A family of functions that intersects another family of functions at right angles

Ex: Find the orthogonal trajectories to the family $x = ky^2$

1. Find the differential equation: $\frac{d}{dx}x = \frac{d}{dx}(ky^2) \Rightarrow 1 = 2ky\frac{dy}{dx}$

$\frac{dy}{dx} = \frac{1}{2ky}$ $\qquad k = \frac{x}{y^2} \Rightarrow \frac{dy}{dx} = \frac{1}{2\frac{x}{y^2}\cdot y} = \frac{y}{2x}$

2. Find the negative reciprocal: $\frac{dy}{dx} = -\frac{2x}{y}$

3. Solve new differential equation for the family of functions:

$$y\,dy = -2x\,dx \Rightarrow \int y\,dy = -2\int x\,dx \Rightarrow \frac{y^2}{2} = -\frac{2x^2}{2} + C$$

$$y^2 = -2x^2 + C \Rightarrow y = \pm\sqrt{C - 2x^2}$$