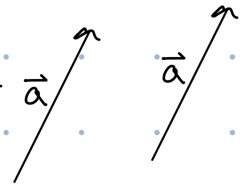
Vectors

Vectors have both a size/magnitude and direction. They can be moved and as long as their size and direction don't change, they're the same vector.

Ex: $\vec{a}$



$\hat{i}$ = x-axis

$\hat{j}$ = y-axis

$\hat{k}$ = z-axis

zero vector $= \vec{0} = 0\hat{i} + 0\hat{j}$

Cartesian representations: $x\hat{i} + y\hat{j}$ or $\langle x, y \rangle$

$r =$ magnitude $= |\vec{a}| = \boxed{a = \sqrt{a_x^2 + a_y^2}}$

Vector addition and subtraction

$$\boxed{\vec{a} + \vec{b} = \langle a_x + b_x ,\ a_y + b_y \rangle}$$

Subtraction: $\vec{a} - \vec{b} = \vec{a} + (-\vec{b})$

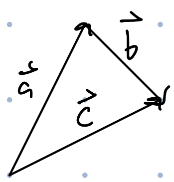
Flip 180°

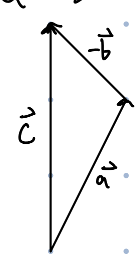
Ex: $\vec{a} = \langle 1, 2 \rangle \quad \vec{b} = \langle 1, -1 \rangle$

$\vec{a} + \vec{b} = \vec{c}$

$\vec{a} - \vec{b} = \vec{c}$

Graphically:





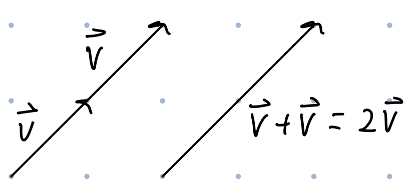
Algebra: $\vec{c} = \langle 1+1, 2-1 \rangle$
$= \langle 2, 1 \rangle$

$\vec{c} = \langle 1-1, 2+1 \rangle$
$= \langle 0, 3 \rangle$

Vector scalar multiplication

Multiplying a scalar to a vector scales that vector. If the scalar is negative, it flips its direction by 180°.

$$\boxed{k\vec{a} = \langle ka_x ,\ ka_y \rangle}$$



Properties of vectors

- $\vec{a} + \vec{b} = \vec{b} + \vec{a}$
- $(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c})$
- $\vec{a} + \vec{0} = \vec{a}$
- $\vec{a} + (-\vec{a}) = \vec{0}$
- $k(\vec{a} + \vec{b}) = k\vec{a} + k\vec{b}$
- $(k + l)\vec{a} = k\vec{a} + l\vec{a}$
- $(kl)\vec{a} = k(l\vec{a})$

Unit vectors

Unit vectors always have a magnitude of 1 and are usualy used to define direction.

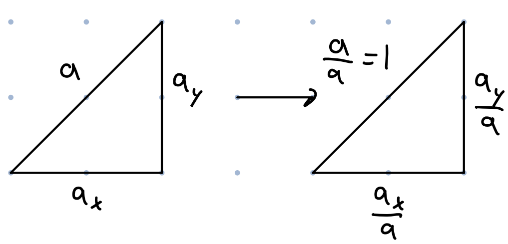
Ex: $\hat{i} = \langle 1, 0, 0 \rangle$
$\hat{j} = \langle 0, 1, 0 \rangle$
$\hat{k} = \langle 0, 0, 1 \rangle$

For 2D: $\langle \cos\theta, \sin\theta \rangle$
3D requires $2\ \theta$s.

To find the unit vector of another vector, divide by it's magnitude.

$$\boxed{\hat{a} = \frac{\vec{a}}{a}} = \frac{a_x\hat{i} + a_y\hat{j}}{\sqrt{a_x^2 + a_y^2}} = \frac{a_x}{\sqrt{a_x^2 + a_y^2}}\hat{i} + \frac{a_y}{\sqrt{a_x^2 + a_y^2}}\hat{j}$$

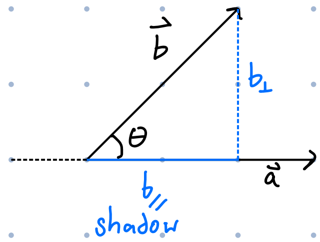
Scale down the vector so it's magnitude is 1



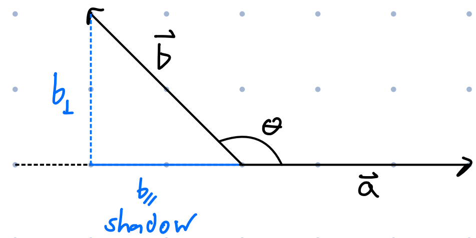
Dot product

The dot product of $\vec{a}$ and $\vec{b}$ is the projection/shadow of $\vec{b}$ onto $\vec{a}$'s line multiplied by $\vec{a}$'s magnitude.

Acute θ:



Obtuse θ:



$$\boxed{\begin{aligned} \vec{a}\cdot\vec{b} &= a\underbrace{b\cos\theta}_{\text{shadow}} \\ \vec{a}\cdot\vec{b} &= a_x b_x + a_y b_y \end{aligned}}$$

$$\text{Comp}_{\vec{a}}(\vec{b}) = \text{length of the shadow} = b_{\parallel} = \frac{\vec{a}\cdot\vec{b}}{a}$$

$$\text{Proj}_{\vec{a}}(\vec{b}) = \text{Shadow's vector} = \vec{b_{\parallel}} = b_{\parallel}\,\hat{a}$$

$$\vec{b} - \vec{b_{\parallel}} = \vec{b_\perp}$$

Properties of dot product

- $\vec{a}\cdot\vec{a} = a^2$
- $\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}$
- $\vec{a}\cdot(\vec{b}+\vec{c}) = \vec{a}\cdot\vec{b} + \vec{a}\cdot\vec{c}$
- $(k\vec{a})\cdot\vec{b} = k(\vec{a}\cdot\vec{b}) = \vec{a}\cdot(k\vec{b})$
- $\vec{a}\cdot\vec{0} = 0$

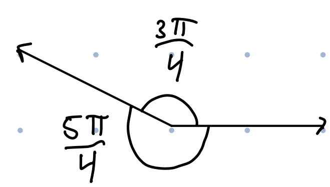
Ex: Find angle between $\langle 3,1\rangle$ and $\langle -2,1\rangle$

$$\vec{a}\cdot\vec{b} = (3)(-2) + (1)(1) = -6+1 = -5$$

$$a = \sqrt{3^2+1^2} = \sqrt{10} \qquad b = \sqrt{(-2)^2+1^2} = \sqrt{5}$$

$$\vec{a}\cdot\vec{b} = ab\cos\theta$$

$$\theta = \cos^{-1}\left(\frac{\vec{a}\cdot\vec{b}}{ab}\right) = \cos^{-1}\left(\frac{-5}{\sqrt{10}\sqrt{5}}\right) = \frac{3\pi}{4} \text{ or } \frac{5\pi}{4}$$



If $\vec{a}\cdot\vec{b} > 0$, then θ is acute.

If $\vec{a}\cdot\vec{b} < 0$, then θ is obtuse.

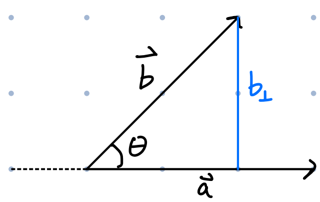
If $\vec{a}\cdot\vec{b} = 0$, then $\theta = \frac{\pi}{2}$ (orthogonal/perpendicular)

Cross Product

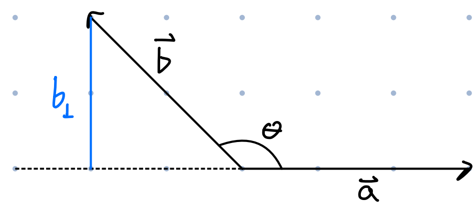
The cross product only works in 3D. The magnitude of $\vec{a} \times \vec{b}$ is $b_\perp a$. It's direction is the right hand rule which is perpendicular to $\vec{a}$ and $\vec{b}$.

$$\boxed{|\vec{a} \times \vec{b}| = a \underbrace{b \sin\theta}_{b_\perp}}$$

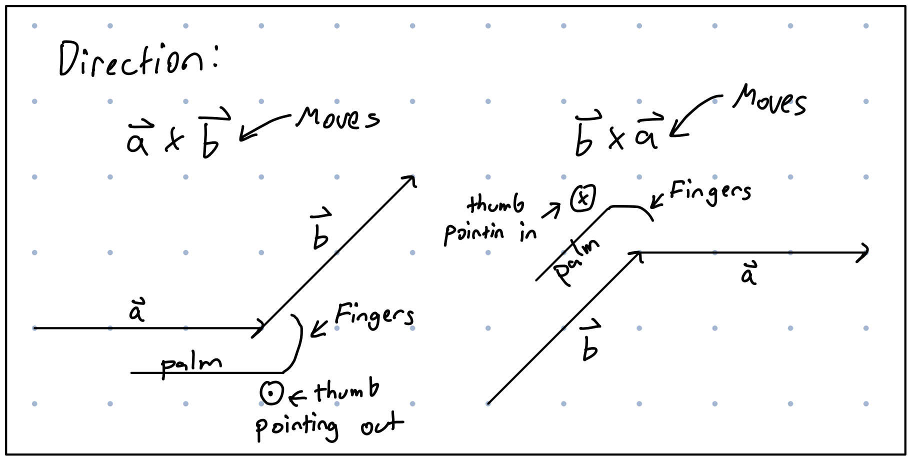
Acute θ:



Obtuse θ:



$$\vec{a} \times \vec{b} = \langle a_y b_z - a_z b_y,\ a_z b_x - a_x b_z,\ a_x b_y - a_y b_x \rangle$$



$$\begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{bmatrix} \begin{matrix} \hat{i} & \hat{j} \\ a_x & a_y \\ b_x & b_y \end{matrix}$$

$$a_y b_z \hat{i} + a_z b_x \hat{j} + a_x b_y \hat{k} -$$
$$a_y b_x \hat{k} - a_z b_y \hat{i} - a_x b_z \hat{j}$$
$$= \langle a_y b_z - a_z b_y,\ a_z b_x - a_x b_z,\ a_x b_y - a_y b_x \rangle$$

Properties of cross product

- $\vec{a} \times \vec{b} = -(\vec{b} \times \vec{a})$
- $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$
- $k(\vec{a} \times \vec{b}) = (k\vec{a}) \times \vec{b} = \vec{a} \times (k\vec{b})$
- $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$
- $(\vec{a} + \vec{b}) \times \vec{c} = \vec{a} \times \vec{c} + \vec{b} \times \vec{c}$
- $\vec{a} \cdot (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \cdot \vec{c}$
- $\vec{a} \times \vec{a} = \vec{0}$

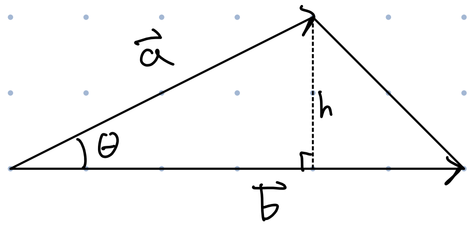
$(\vec{a} \times b) \perp \vec{a}$

and

$(\vec{a} \times \vec{b}) \perp \vec{b}$

$\vec{a} \cdot (\vec{a} \times \vec{b}) = 0$

$\vec{b} \cdot (\vec{a} \times \vec{b}) = 0$



$h = a\sin\theta$

$$A = \frac{1}{2}bh = \frac{1}{2}ba\sin\theta = \frac{|\vec{b}\times\vec{a}|}{2}$$

Find the area of the $\triangle$ with vertices:

$(1,1,1),\ (2,1,0),\ (0,3,1)$

$$\vec{a} = \langle 1-2,\ 1-1,\ 1-0\rangle = \langle 1, 0, 1\rangle$$

$$\vec{b} = \langle 1-0,\ 1-3,\ 1-1\rangle = \langle 1, -2, 0\rangle$$

$$\begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 1 \\ 1 & -2 & 0 \end{bmatrix} \begin{matrix} \hat{i} & \hat{j} \\ 1 & 0 \\ 1 & -2 \end{matrix}$$

$$(0)(0)\hat{i} + (1)(1)\hat{j} + (1)(-2)\hat{k} -$$

$$(0)(1)\hat{k} - (1)(-2)\hat{i} - (0)(1)\hat{j}$$

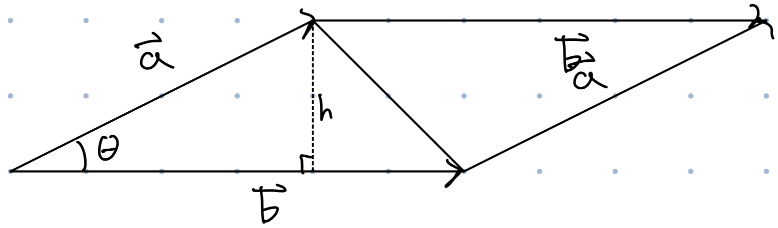
$$= (0+2)\hat{i} + (1-0)\hat{j}\ (-2-0)\hat{k}$$

$$\vec{a}\times\vec{b} = 2\hat{i} + \hat{j} - 2\hat{k}$$

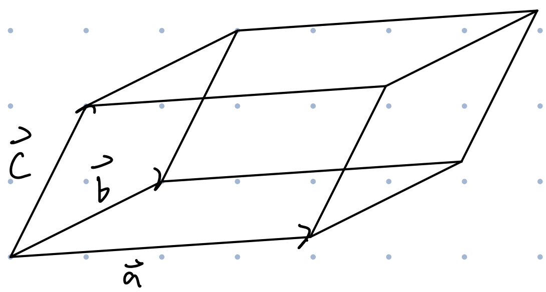
$$|\vec{a}\times\vec{b}| = \sqrt{2^2 + 1^2 + (-2)^2} = \sqrt{4+1+4} = \sqrt{9} = 3$$

$$\boxed{A = \frac{3}{2}}$$

Vector from A to B is B − A



Area of parallelagram = $|\vec{a} \times \vec{b}|$



Determinant

Volume of parallelepiped = $\left|\vec{a} \cdot (\vec{b} \times \vec{c})\right| = \begin{bmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{bmatrix}$