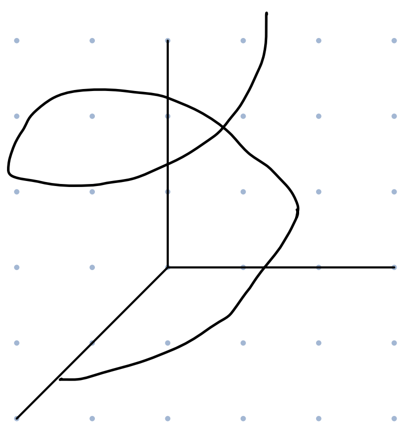
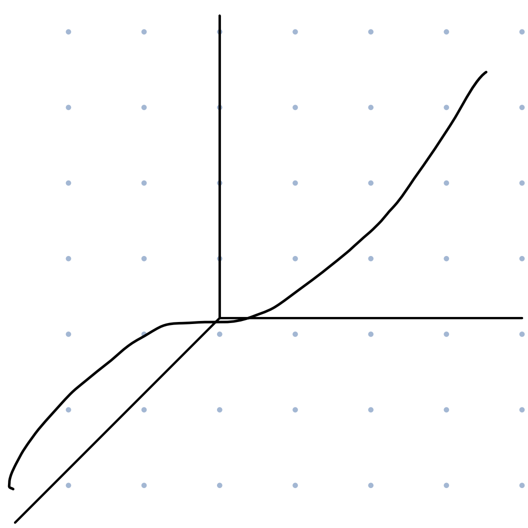
Vector valued functions

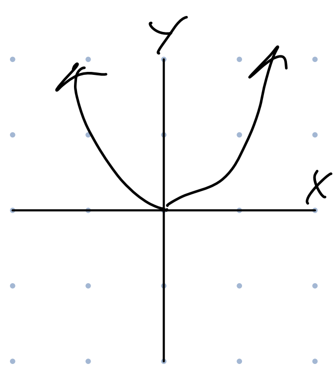
Functions with real number inputs and vectors as output.
The tips sweep out a curve in 3D space.

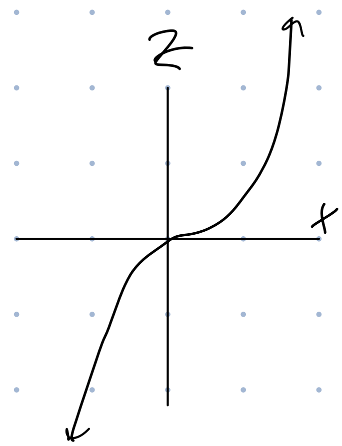
Ex: The standard helix: $\vec{r}(t) = \langle \cos t, \sin t, t \rangle$



Ex: The twisted cubic: $\vec{r}(t) = \langle t, t^2, t^3 \rangle$







$$\boxed{\lim_{x \to a} \vec{r}(t) = \left\langle \lim_{x \to a} f(t),\ \lim_{x \to a} g(t),\ \lim_{x \to a} h(t) \right\rangle}$$

$$\boxed{\vec{r}\,'(t) = \langle f'(t), g'(t), h'(t) \rangle}$$

$$\boxed{\int \vec{r}(t) = \left\langle \int f(t)\,dt,\ \int g(t)\,dt,\ \int h(t)\,dt \right\rangle}$$

Properties of derivatives

$\vec{U}(t)$ and $\vec{V}(t)$ are vector-valued functions.

c is a scalar

$f(t)$ is a scalar-valued function.

- $\frac{d}{dt}\left(\vec{U}(t) + \vec{V}(t)\right) = \vec{U}'(t) + \vec{V}'(t)$

- $\frac{d}{dt}\left(c\vec{U}(t)\right) = c\vec{U}'(t)$

- $\frac{d}{dt}\left(f(t)\vec{U}(t)\right) = f'(t)\vec{U}(t) + f(t)\vec{U}'(t)$

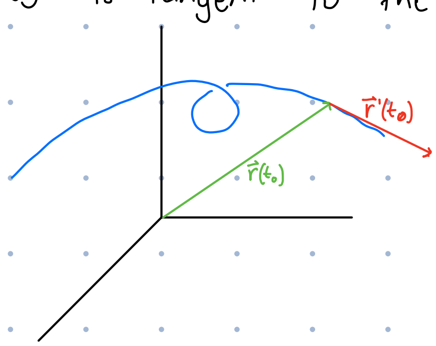
- $\frac{d}{dt}\left(\vec{U}(t) \cdot \vec{V}(t)\right) = \vec{U}'(t) \cdot \vec{V}(t) + \vec{U}(t)\vec{V}'(t)$

- $\frac{d}{dt}\left(\vec{U}(t) \times \vec{V}(t)\right) = \vec{U}'(t) \times \vec{V}(t) + \vec{U}(t) \times \vec{V}'(t)$

product rule

- $\frac{d}{dt}\left(\vec{U}(f(t))\right) = f'(t)\vec{U}'(f(t))$ — Chain rule

$\vec{r}\,'(t_0)$ is tangent to the curve $\vec{r}(t)$ at t_0 as long as $\vec{r}(t)$ is smooth



Smooth

1) Continuous / No gaps $\left(\lim_{t \to a} \vec{r}(t) = \vec{r}(a)\right)$.
2) $\vec{r}(t)$ is differentiable (No sharp corners).
3) $\vec{r}\,'(t)$ is never $\vec{0}$ (Not moving over t).

Unit vectors

Tangent unit vector: $\vec{T} = \dfrac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}$ Same direction as t increases

Normal unit vector: $\vec{N} = \dfrac{\vec{T}\,'(t)}{|\vec{T}\,'(t)|} = \vec{B} \times \vec{T}$

Binormal unit vector: $\vec{B} = \vec{T} \times \vec{N} = \dfrac{\vec{r}\,'(t) \times \vec{r}\,''(t)}{|\vec{r}\,'(t) \times \vec{r}\,''(t)|}$

- Normal plane is $\perp$ to $\vec{T}$.
- Rectifying plane is $\perp$ to $\vec{N}$.
- Osculating plane is $\perp$ to $\vec{B}$.

Ex: Find $\vec{T}$, $\vec{N}$, & $\vec{B}$ at the point $(1,1,1)$ on the twisted cubic.

	At point	
$\vec{r}(t) = \langle t, t^2, t^3 \rangle$	$\langle 1, 1, 1 \rangle$	$\lvert\vec{r}(1)\rvert = \sqrt{14}$
$\vec{r}\,'(t) = \langle 1, 2t, 3t^2 \rangle$	$\langle 1, 2, 3 \rangle$	
$\vec{r}\,''(t) = \langle 0, 2, 6t \rangle$	$\langle 0, 2, 6 \rangle$	

$\vec{r}\,'(1) \times \vec{r}\,''(1) = \langle 6, -6, 2 \rangle = 2\langle 3, -3, 1 \rangle$ $\quad |\vec{r}\,'(1) \times \vec{r}\,''(1)| = 2\sqrt{19}$

$\vec{T} = \left\langle \frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}} \right\rangle$

$\vec{B} = \left\langle \frac{3}{\sqrt{19}}, \frac{-3}{\sqrt{19}}, \frac{1}{\sqrt{19}} \right\rangle$

$\vec{N} = \vec{B} \times \vec{T} = \left\langle \frac{-11}{\sqrt{266}}, \frac{-8}{\sqrt{266}}, \frac{9}{\sqrt{266}} \right\rangle$

Useful fact

If $|\vec{r}(t)|$ is constant, then $\vec{r}(t) \perp \vec{r}\,'(t)$.

Proof:

If $|\vec{r}(t)|$ is constant, then $|\vec{r}(t)|^2$ is constant.

So $\frac{d}{dt}\left(|\vec{r}(t)|^2\right) = 0$

$\rightarrow \frac{d}{dt}\left(\vec{r}(t)\cdot\vec{r}(t)\right) = 0$

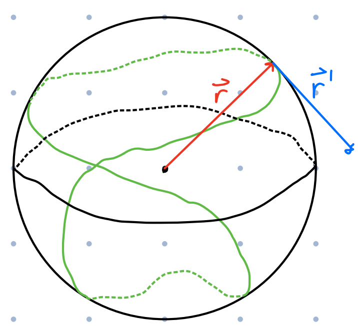
$\rightarrow \vec{r}\,'(t)\cdot\vec{r}(t) + \vec{r}(t)\cdot\vec{r}\,'(t) = 0$

$\rightarrow 2\left(\vec{r}(t)\cdot\vec{r}\,'(t)\right) = 0$

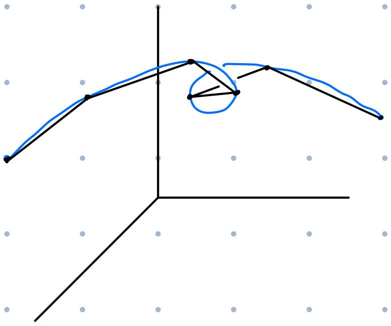
$\rightarrow \vec{r}(t)\cdot\vec{r}\,'(t) = 0$

$\therefore \vec{r}(t) \perp \vec{r}\,'(t)$

If $|\vec{r}(t)| = k$, then $\vec{r}(t)$ is on the sphere of radius k centered at $(0,0,0)$.



Arc Length



Approximate using tangent vectors.

$$\boxed{\text{Arc length} = \int_a^b |\vec{r}\,'(t)|\,dt}$$

- It can be transformed into a function when given a starting point.

$$s(t) = \int_a^t |\vec{r}\,'(u)|\,du$$

Ex: Find $s(t)$ for $\vec{r}(t) = \langle \cos t, \sin t, t \rangle$ at the starting point $(1, 0, 0)$.
$t = 0$

$\vec{r}\,'(t) = \langle -\sin t, \cos t, 1 \rangle$

$\vec{r}\,'(0) = \langle 0, 1, 1 \rangle$

$|\vec{r}\,'(0)| = \sqrt{2}$

$$s(t) = \int_0^t \sqrt{2}\,du = \sqrt{2}\,t$$

- You can re-parameterize $\vec{r}(t)$ in terms of s.

Ex: $s(t) = \sqrt{2}\,t$

$t = \frac{s}{\sqrt{2}}$

$$\vec{r}(s) = \left\langle \cos\left(\frac{s}{\sqrt{2}}\right), \sin\left(\frac{s}{\sqrt{2}}\right), \frac{s}{\sqrt{2}} \right\rangle$$

Curvature

How curvey is a function. Small when it's strait and large when curvy.

$$\kappa = \left| \frac{d\vec{T}(s)}{ds} \right|$$

chain rule

$$\frac{d\vec{T}}{dt} = \frac{d\vec{T}}{ds} \cdot \frac{ds}{dt} \rightsquigarrow \frac{d\vec{T}}{ds} = \frac{d\vec{T}/dt}{ds/dt} = \frac{\vec{T}'(t)}{|\vec{r}'(t)|}$$

$$\frac{d}{dt} s(t) = \frac{d}{dt} \int_a^t |\vec{r}'(u)|\,du$$

$$\kappa = \frac{|\vec{T}'(t)|}{|\vec{r}'(t)|}$$

$$\boxed{\kappa = \frac{|\vec{r}'(t) \times \vec{r}''(t)|}{|\vec{r}'(t)|^3}}$$

$$\vec{T}(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|} \to \vec{r}' = |\vec{r}'|\vec{T}$$

Product rule

$$\to \vec{r}'' = \frac{d}{dt}|\vec{r}'|\vec{T} + |\vec{r}'|\vec{T}'$$

$$\vec{r}' \times \vec{r}'' = \left[|\vec{r}'|\vec{T} \times \frac{d}{dt}|\vec{r}'|\vec{T}\right] + \left[|\vec{r}'|\vec{T} \times |\vec{r}'|\vec{T}'\right] = \left[|\vec{r}'| \frac{d}{dt}|\vec{r}'|\right] \overbrace{\cancel{\vec{T} \times \vec{T}}}^{0} + \left[|\vec{r}'|^2\right] \vec{T} \times \vec{T}'$$

$$|\vec{r}' \times \vec{r}''| = \left| |\vec{r}'|^2 (\vec{T} \times \vec{T}') \right|$$

$$|\vec{r}' \times \vec{r}''| = \left| |\vec{r}'|^2 \overbrace{\cancel{|\vec{T}|}}^{1} |\vec{T}'| \overbrace{\cancel{\sin 90^\circ}}^{1} \right|$$

since $|\vec{T}'|$ is constant, $T \perp T'$

$$|\vec{T}'| = \frac{|\vec{r}' \times \vec{r}''|}{|\vec{r}'|^2}$$

Ex: Find κ for $\vec{r}(t) = \langle a\cos t, a\sin t, 0 \rangle$ a circle of radius a.

$\vec{r}'(t) = \langle -a\sin t, a\cos t, 0 \rangle$ $\quad |\vec{r}'(t)| = a$

$\vec{r}''(t) = \langle -a\cos t, -a\sin t, 0 \rangle$

$\vec{r}'(t) \times \vec{r}''(t) = \langle 0, 0, a^2 \rangle$ $\quad |\vec{r}'(t) \times \vec{r}''(t)| = a^2$

$$\boxed{\kappa = \frac{1}{a} \leftarrow \text{radius of circle}}$$

κ for $f(x)$ in xy-plane

$$y = f(x) \quad \hookrightarrow \quad \vec{r}(t) = \langle t, f(t), 0 \rangle$$

$$\vec{r}\,'(t) = \langle 1, f'(t), 0 \rangle \qquad |\vec{r}\,'(t)| = \sqrt{1 + (f'(t))^2}$$

$$\vec{r}\,''(t) = \langle 0, f''(t), 0 \rangle$$

$$\vec{r}\,'(t) \times \vec{r}\,''(t) = \langle 0, 0, f''(t) \rangle \qquad |\vec{r}\,'(t) \times \vec{r}\,''(t)| = |f''(t)|$$

$$\boxed{\kappa_{f(x)} = \frac{|f''(t)|}{\left(1 + (f'(t))^2\right)^{3/2}}}$$

← Absolute value

Motion in space

Velocity $= \vec{v}(t) = \vec{r}\,'(t)$

Acceleration $= \vec{a}(t) = \vec{v}\,'(t) = \vec{r}\,''(t)$

Speed $= |\vec{v}(t)| = |\vec{r}\,'(t)|$

$$|\vec{a}_T| = \frac{\vec{v} \cdot \vec{a}}{|\vec{v}|}$$

$$|\vec{a}_N| = \frac{|\vec{v} \times \vec{a}|}{|\vec{v}|}$$