Sequences

$a_1, a_2, a_3, \ldots, a_n$

Arithmetic: $a_1 \xrightarrow{\pm d} a_2 \xrightarrow{\pm d} a_3, \ldots$

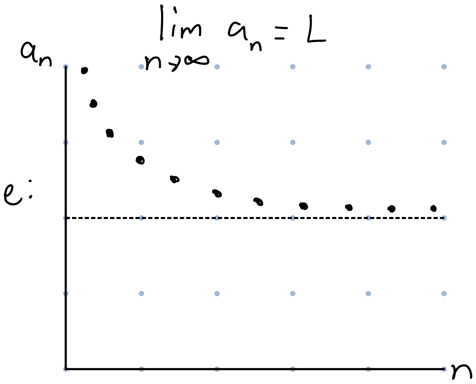
Geometric: $a_1 \xrightarrow{\cdot r} a_2 \xrightarrow{\cdot r} a_3, \ldots$

Ex: $-\frac{1}{2}, \frac{2}{3}, -\frac{3}{4}, \frac{4}{5}, -\frac{5}{6}, \ldots$ $\qquad a_n = (-1)^n \frac{n}{n+1}$

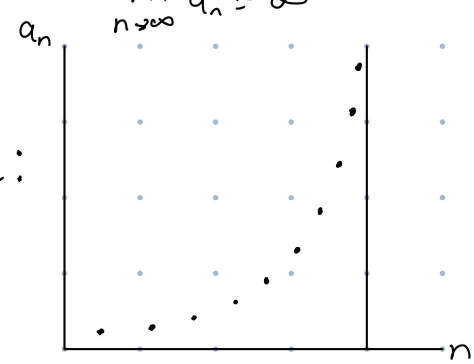
$n = 1, 2, 3, 4, 5$

Ex: $\frac{3}{4}, \frac{9}{7}, \frac{27}{10}, \frac{81}{13}, \frac{243}{16}, \ldots$ $\qquad a_n = \frac{3^n}{3n+1}$

Converge: $\lim_{n\to\infty} a_n = L$



Diverge: $\lim_{n\to\infty} a_n = \infty$



Property of sequences

1) $\lim_{n\to\infty} C = C$
2) $\lim_{n\to\infty} Ca_n = C \cdot \lim_{n\to\infty} a_n = CA$
3) $\lim_{n\to\infty} (a_n \pm b_n) = \lim_{n\to\infty} a_n \pm \lim_{n\to\infty} b_n = A \pm B$
4) $\lim_{n\to\infty} (a_n \cdot b_n) = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n = AB$
5) $\lim_{n\to\infty} \left(\frac{a_n}{b_n}\right) = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n} = \frac{A}{B}$ where $B \neq 0$
6) $\lim_{n\to\infty} (a_n)^P = A^P$ where $P > 0$ and $a_n > 0$

Ex: $\lim_{n\to\infty} \left(1+\frac{4}{n}\right)^n = \lim_{n\to\infty} e^{\ln\left(1+\frac{4}{n}\right)^n}$

$= e^{\lim_{n\to\infty} \ln\left(1+\frac{4}{n}\right)^n} = e^{\lim_{n\to\infty} n \ln\left(\frac{n+4}{n}\right)}$

$= e^{\lim_{n\to\infty} \frac{\ln(n+4) - \ln(n)}{1/n}} = e^{\lim_{n\to\infty} \frac{\frac{1}{n+4} - \frac{1}{n}}{-\frac{1}{n^2}}}$

$= e^{\lim_{n\to\infty} \frac{n-(n+4)}{n(n+4)} \cdot -n^2} = e^{\lim_{n\to\infty} \frac{-4}{n(n+4)} \cdot -n^2}$

$= e^{\lim_{n\to\infty} \frac{4n}{n+4}} = e^{\lim_{n\to\infty} \frac{4}{1}} = e^4$

Ex: $a_n = \frac{\cos n}{2^n}$ converges?

$$-1 \leq \cos n \leq 1$$

$$\frac{-1}{2^n} \leq \frac{\cos n}{2^n} \leq \frac{1}{2^n}$$

$$\lim_{n\to\infty} -\frac{1}{2^n} \leq \lim_{n\to\infty} \frac{\cos n}{2^n} \leq \lim_{n\to\infty} \frac{1}{2^n} \qquad 0 \leq \lim_{n\to\infty} \frac{\cos n}{2^n} \leq 0$$

a_n converges to 0 by the squeeze theorem.

Monotone - Sequence is increasing or decreasing.

Bounded - There's a constant $M > a_n$ and $a_n > m$.

Monotone convergence theorem

If a sequence is monotone and bounded, then it's convergent.

Ex: Is $a_n = \frac{1}{2n+3}$ convergent? Yes

Monotone?

$$a_{n+1} = \frac{1}{2(n+1)+3} = \frac{1}{2n+2+3} = \frac{1}{2n+5}$$

$a_n > a_{n+1}$ It's decreasing so it's monotone.

Bounded?

Since it's decreasing $a_1 = \frac{1}{5} = M$ bounds a_n above.

Since it's decreasing $\lim_{n\to\infty} a_n = 0 = m$ bounds a_n below.

Overview

Tests

1. Test for divergence.

Non-alternating

4. Integral test
4. Direct comparison test
2. Limit comparison test

Alternating

3. Alternating series test

Both

2. Ratio test
2. Root test

Series

- Geometry series
- P-series
- Power series
- Taylor series/Maclaurin series

Series

Finite: $S_k = \sum_{n=1}^{k} a_n = a_1 + a_2 + \ldots + a_k$

Infinte: $S = \sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \ldots$

$$\sum_{i=1}^{n} 1 = n$$

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$$

Geometry Series: $\sum_{n=1}^{\infty} ar^{n-1}$

If $|r| < 1$, then $\sum_{n=1}^{\infty} ar^{n-1} = \frac{a_1}{1-r}$

If $|r| \geq 1$, then it diverges.

Ex: $\sum_{n=1}^{\infty} \frac{(-3)^{n+1}}{4^{n-1}} = \sum_{n=1}^{\infty} \frac{(-3)^{n-1+2}}{4^{n-1}} = \sum_{n=1}^{\infty} \frac{(-3)^{n-1}(-3)^2}{4^{n-1}} = \sum_{n=1}^{\infty} 9\left(-\frac{3}{4}\right)^{n-1}$

$\left|-\frac{3}{4}\right| < 1 : a_1 = 9 \quad S = \sum_{n=1}^{\infty} 9\left(-\frac{3}{4}\right)^{n-1} = \frac{9}{1+\frac{3}{4}} = \frac{36}{7}$

Ex: $\sum_{n=1}^{\infty} \frac{e^n}{3^{n+1}} = \sum_{n=1}^{\infty} \frac{e^n}{3^n \cdot 3} = \sum_{n=1}^{\infty} \frac{1}{3}\left(\frac{e}{3}\right)^n$

$$\frac{e}{3^2} + \frac{e^2}{3^3} + \frac{e^3}{3^4} + \ldots$$

$\cdot \frac{e}{3} \qquad \cdot \frac{e}{3}$

$\left|\frac{e}{3}\right| < 1 : a_1 = \frac{e}{9} \qquad S = \frac{\frac{e}{9}}{1-\frac{e}{3}} = \frac{e}{3(3-e)}$

Telescoping series: Terms cancel so you're just left with some of the first terms and some of the last.

Ex: $\sum_{n=1}^{\infty} \ln\left(\frac{n}{n+1}\right) = \lim_{n\to\infty} \sum_{i=1}^{n} \ln\left(\frac{i}{i+1}\right) = \lim_{n\to\infty} \sum_{i=1}^{n} \ln i - \ln(i+1)$

$= (\ln 1 - \cancel{\ln 2}) + (\cancel{\ln 2} - \cancel{\ln 3}) + \ldots + (\cancel{\ln n} - \ln(n+1)) = \overset{0}{\cancel{\ln 1}} - \ln(n+1)$

Harmonic series: $\sum_{n=1}^{\infty} \frac{1}{n}$ Always diverges even when $\lim_{n\to\infty} a_n = 0$.

$$1 + \frac{1}{2} + \underbrace{\frac{1}{3} + \frac{1}{4}}_{>\frac{2}{4}} + \underbrace{\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}}_{>\frac{4}{8}} + \underbrace{\frac{1}{9} + \dots + \frac{1}{16}}_{>\frac{8}{16}} + \dots$$

$S_1 = 1$

$S_2 = 1.5$

$S_4 > 2$

$S_8 > 2.5$

$S_{16} > 3$

$\vdots$

$\lim_{n\to\infty} S_n = \infty$ Diverges

Test for divergence

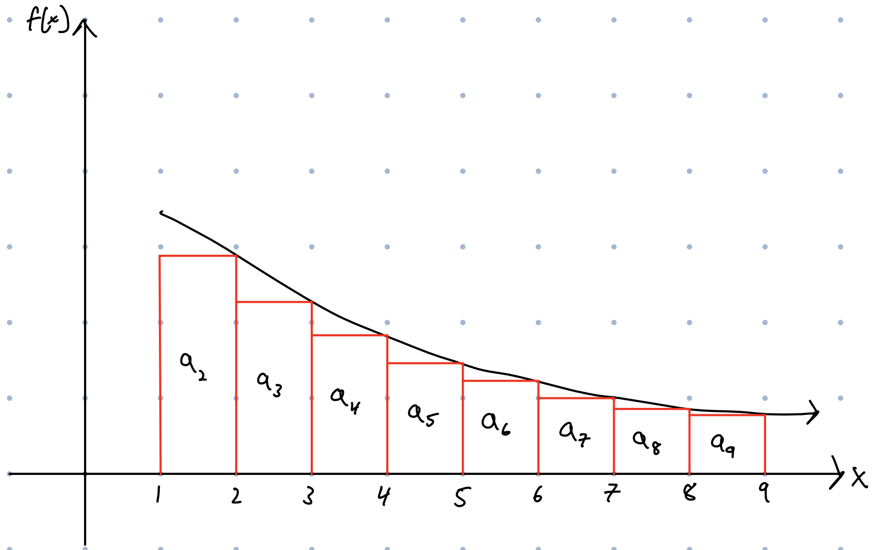
$$\boxed{\text{If } \lim_{n\to\infty} a_n \neq 0 \text{ or DNE, then } \sum_{n=1}^{\infty} a_n \text{ diverges.}}$$

Ex: Does $\sum_{n=1}^{\infty} \frac{n^2+2}{2n^2+1}$ converge or diverge?

$$\lim_{n\to\infty} \frac{n^2+2}{2n^2+1} = \frac{1}{2} \quad \text{so} \quad \sum_{n=1}^{\infty} \frac{n^2+2}{2n^2+1} = a_1 + a_2 + \dots + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \dots = \infty$$

The integral test

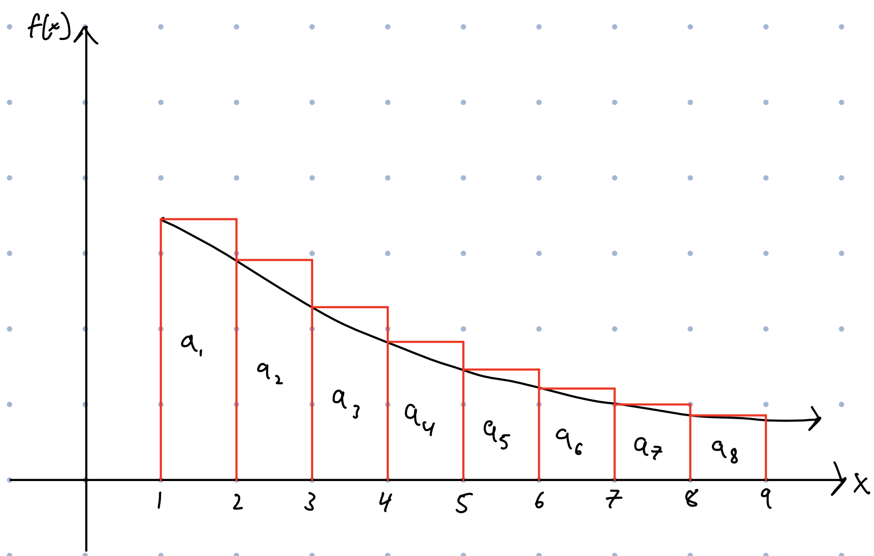
$f(x)$ is continious, positive, and decreasing from $[1, \infty)$ and $f(n) = a_n$



Since the width is 1, the area is the hight.

$$\sum_{n=1}^{\infty} a_n \leq a_1 + \int_1^{\infty} f(x)\,dx$$

$$\boxed{\text{If } \int_1^{\infty} f(x)\,dx \text{ converges, then } \sum_{n=1}^{\infty} a_n \text{ converges.}}$$



$$\sum_{n=1}^{\infty} a_n \geq \int_1^{\infty} f(x)\,dx$$

$$\boxed{\text{If } \int_1^{\infty} f(x)\,dx \text{ diverges, then } \sum_{n=1}^{\infty} a_n \text{ diverges.}}$$

P-series : $\boxed{\text{Converges when } p>1}$

What values of P does $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converge?

$p<0$: (P is negative)

$\lim_{n\to\infty} \frac{1}{n^{-p}} = \lim_{n\to\infty} n^p = \infty$ so diverges by test for divergence.

$p=0$:

$\lim_{n\to\infty} \frac{1}{n^0} = 1$ so diverges by test for divergence.

$p>0$ & $p\neq 1$:

Let $f(t) = \frac{1}{x^p}$

From $[1,\infty)$

continious ✓

positive ✓

decreasing ✓

$\int_1^{\infty} \frac{1}{x^p}dx = \frac{1}{1-p} x^{1-p} \Big|_1^{\infty}$ when $p>1$, $\int_1^{\infty} \frac{1}{x^p}dx = \frac{1}{1-p}$

So it converges by the integral test.

$P=1$: Harmonic series which diverges.

Direct comparison test

$\emptyset < a_n \leq b_n$

- If $\sum b_n$ converges, then $\sum a_n$ converges.
- If $\sum a_n$ diverges, then $\sum b_n$ diverges.

Ex: Does $\sum_{n=1}^{\infty} \frac{3}{n+1}$ converge?

Compare it with $\frac{1}{n}$:

For $n \geq 1$:

$n+1 < 3n$

$\frac{n+1}{n} < 3$

$\frac{1}{n+1} \cdot \frac{n+1}{n} < 3 \cdot \frac{1}{n+1}$

$\frac{1}{n} < \frac{3}{n+1}$

$\sum_{n=1}^{\infty} \frac{1}{n}$ diverges so $\sum_{n=1}^{\infty} \frac{3}{n+1}$ also diverges.

by the direct comparison test.

Limit comparison test

$a_n, b_n > \emptyset$

- If $\lim_{n\to\infty} \frac{a_n}{b_n} = L \neq \emptyset$

then, $\sum a_n$ and $\sum b_n$ both either converge or diverge.

Ex: Does $\sum_{n=1}^{\infty} \frac{n+1}{n^3-n}$ converge?

$$\lim_{n\to\infty} \frac{\frac{n+1}{n^3-n}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^3+n^2}{n^3-n} = 1 \neq \emptyset$$

$\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges so it converges.

Alternating series test

If $\sum a_n$ is an alternating series (alternating pos/neg), and if

- $|a_n|$ is decreasing

- $\lim\limits_{n\to\infty} |a_n| = 0$

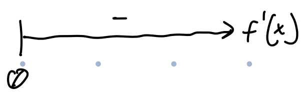
then $\sum a_n$ converges.

Ex: Does $\sum\limits_{n=1}^{\infty} (-1)^{n+1}\left(\frac{1}{n}\right)$ converge?

- Is $\left|(-1)^{n+1}\left(\frac{1}{n}\right)\right| = \frac{1}{n}$ decreasing?

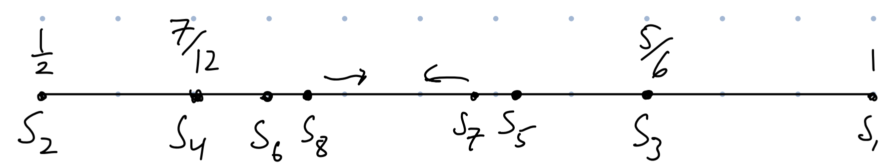
$f(x) = \frac{1}{x}$

$f'(x) = -\frac{1}{x^2} \to \cancel{0 = -\frac{1}{x^2}}$ $\quad$ DNE $= -\frac{1}{x^2} \to 0 = x^2 \quad x = 0$



Yes decreasing ✓

- $\lim\limits_{n\to\infty} \frac{1}{n} = 0$ ✓



Yes it converges.

3 options for alternating series (a_n).

1. **Converges absolutely** - If $\sum |a_n|$ converges, then $\sum a_n$ also converges.

2. **Converges conditionally** - $\sum |a_n|$ diverges, but $\sum a_n$ converges.

3. **Diverges** - $\sum a_n$ diverges (Can use test for divergence).

Ex: Which option is $\sum_{n=1}^{\infty} \left(\frac{(-1)^{n+1}\sqrt{n}}{n+1} \right)$?

- Does $\sum_{n=1}^{\infty} \left| \frac{(-1)^{n+1}\sqrt{n}}{n+1} \right| = \sum_{n=1}^{\infty} \frac{\sqrt{n}}{n+1}$ converge?

$$\lim_{n\to\infty} \frac{\frac{\sqrt{n}}{n+1}}{\frac{1}{\sqrt{n}}} = \lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} \frac{\frac{n}{n}}{\frac{n}{n}+\frac{1}{n}} = 1 \neq 0$$

$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges so it diverges.

It's doesn't converge absolutely.

- Is $\lim_{n\to\infty} |a_n| = 0$?

$$\lim_{n\to\infty} \frac{\sqrt{n}}{n+1} = \lim_{n\to\infty} \frac{\frac{1}{2\sqrt{n}}}{1} = 0 \checkmark \text{ Yes}$$

- Is $|a_n|$ decreasing?

$$f(x) = \frac{\sqrt{x}}{x+1} \qquad f'(x) = \frac{(x+1)\left(\frac{1}{2\sqrt{x}}\right) - (1)(\sqrt{x})}{(x+1)^2} = \frac{\frac{x}{2\sqrt{x}} + \frac{1}{2\sqrt{x}} - \sqrt{x}}{(x+1)^2} = \frac{\frac{x+1-2x}{2\sqrt{x}}}{(x+1)^2}$$

$$= \frac{1-x}{2\sqrt{x}(x+1)^2} \rightarrow 0 = 1-x \rightarrow x = 1$$

$$0 = 2\sqrt{x}(x+1)^2 \rightarrow 0 = \sqrt{x} \rightarrow x = 0$$

$$\rightarrow 0 = (x+1)^2 \rightarrow x = -1$$

Yes decreasing ✓

$$1 \; \overset{-}{\longrightarrow} \; f'(x)$$

So it's conditionally convergent.

- If $\sum a_n$ converges absolutely every re-arrangement of $\sum a_n$ converges to the same value.
- If $\sum b_n$ converges conditionally there is a rearragement of $\sum b_n$ that converges to any real number.

Ratio test

- If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1$, $\sum a_n$ converges absolutely.
- If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| > 1$ (or $= \infty$), $\sum a_n$ diverges.
- If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$, No info given.

Ex: $\sum_{n=1}^{\infty} \frac{n^{100}}{6^n}$

$$\lim_{n\to\infty}\left|\frac{\left(\frac{(n+1)^{100}}{6^{n+1}}\right)}{\left(\frac{n^{100}}{6^n}\right)}\right| = \lim_{n\to\infty}\left|\frac{(n+1)^{100}\,6^n}{n^{100}\,6^{n+1}}\right| = \lim_{n\to\infty}\left|\frac{(n+1)^{100}}{6n^{100}}\right| = \frac{1}{6} < 1$$

Converges absolutely by the ratio test.

Root test

- If $\lim_{n\to\infty} \sqrt[n]{|a_n|} < 1$, $\sum a_n$ converges absolutely.
- If $\lim_{n\to\infty} \sqrt[n]{|a_n|} > 1$ (or $= \infty$), $\sum a_n$ diverges.
- If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1$, No info given.

Ex: $$\sum_{n=1}^{\infty} \left(\frac{5^{2n+1}}{n^n}\right)$$

$$\lim_{n\to\infty} \sqrt[n]{\frac{5^{2n+1}}{n^n}} = \lim_{n\to\infty} \frac{\sqrt[n]{25^n \cdot 5}}{n} = \lim_{n\to\infty} \frac{25\sqrt[n]{5}}{n}$$

$$= \lim_{n\to\infty} \frac{25(5)^{\frac{1}{n}}}{n} = 0 < 1$$

Converges absolutely by the root test.

Ex 1: $\displaystyle\sum_{k=2}^{\infty} \frac{k\ln k}{(k+1)^3}$

$\ln k < \sqrt{k}$

$k\ln k < k\sqrt{k}$

$\dfrac{k\ln k}{(k+1)^2} < \dfrac{k^{3/2}}{(k+1)^3}$

By the Direct Comparison test, if $\displaystyle\sum_{k=2}^{\infty} \frac{k^{3/2}}{(k+1)^3}$ converges, the $\displaystyle\sum_{k=2}^{\infty} \frac{k\ln k}{(k+1)^3}$ converges.

Limit comparison test:

$$\lim_{k\to\infty} \left| \frac{\frac{k^{3/2}}{(k+1)^3}}{\frac{1}{k^{3/2}}} \right| = \lim_{k\to\infty} \left| \frac{k^3}{(k+1)^3} \right| = 1 \neq \emptyset$$

$\displaystyle\sum_{k=2}^{\infty} \frac{1}{k^{3/2}}$ converges bc it's a p-series where $p = \frac{3}{2} > 1$

Ex 2: $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^{n^2}$

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n = \lim_{n\to\infty} e^{\ln\left(\frac{n}{n+1}\right)^n} = e^{\lim_{n\to\infty} n \ln\left(\frac{n}{n+1}\right)}$$

$$\lim_{n\to\infty} n\ln\left(\frac{n}{n+1}\right) = \lim_{n\to\infty} \frac{\ln\left(\frac{n}{n+1}\right)}{\frac{1}{n}} \xrightarrow{\text{L'Hopitals}} \lim_{n\to\infty} \frac{\frac{1}{\frac{n}{n+1}} \cdot \frac{(n+1)(1)-(1)(n)}{(n+1)^2}}{-\frac{1}{n^2}} = \lim_{n\to\infty} \frac{\frac{n+1}{n} \cdot \frac{1}{(n+1)^2}}{-\frac{1}{n^2}}$$

$$= \lim_{n\to\infty} \frac{\frac{1}{n(n+1)}}{-\frac{1}{n^2}} = \lim_{n\to\infty} -\frac{n^2}{n(n+1)} = \lim_{n\to\infty} -\frac{n}{n+1} = -1$$

$e^{-1} < 1$ so it converges by root test.

Power series

$$f(x) = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \dots$$

- The domain of $f(x)$ is all the x values which the power series converges.
- It always converges when $x = 0$.

3 possibilities — Radius of convergence (R)

1. It converges absolutely for every value of x. — $R = \infty$
2. Only converges when $x = 0$. — $R = 0$
3. Converges absolutely when $|x| < R$
   - It may or may not converge when $x = R$ or $x = -R$.

Ex 1: Find domain for $\displaystyle\sum_{n=0}^{\infty} \frac{n^3}{(-3)^n} x^n$

Ratio test:

$$\lim_{n\to\infty} \left| \frac{\frac{(n+1)^3 x^{n+1}}{(-3)^{n+1}}}{\frac{n^3 x^n}{(-3)^n}} \right| = \lim_{n\to\infty} \left| \frac{(n+1)^3 x^n x}{(-3)^n(-3)} \cdot \frac{(-3)^n}{n^3 x^n} \right| = \lim_{n\to\infty} \left| \frac{(n+1)^3 x}{-3n^3} \right| = \frac{|x|}{3} < 1 \text{ in order to converge}$$

$|x| < 3$

$-3 < x < 3$

Check endpoints:

$$\sum_{n=0}^{\infty} \frac{n^3 (-3)^n}{(-3)^n} = \sum_{n=0}^{\infty} n^3 \to \text{Diverges}$$

$$\sum_{n=0}^{\infty} \frac{n^3 (3)^n}{(-3)^n} = \sum_{n=0}^{\infty} (-1)^n n^3 \to \text{Diverges}$$

Due to test for divergence.

Domain: $(-3, 3)$

Ex 2: $\displaystyle\sum_{n=1}^{\infty} \frac{(x-7)^n}{n^3 \cdot (-2)^n}$

ratio test:

$$\lim_{n\to\infty} \left| \frac{(x-7)^{n+1}}{(n+1)^3 \cdot (-2)^{n+1}} \cdot \frac{n^3(-2)^n}{(x-7)^n} \right| = \lim_{n\to\infty} \left| \frac{(x-7)\,n^3}{(n+1)^3\,(-2)} \right| = \frac{|x-7|}{2} < 1$$

$$|x-7| < 2$$

$$-2 < x-7 < 2 \longrightarrow 5 < x < 9$$

check endpoints:

$$\sum_{n=1}^{\infty} \frac{(5-7)^n}{n^3 \cdot (-2)^n} = \sum_{n=1}^{\infty} \frac{(-2)^n}{n^3(-2)^n} = \sum_{n=1}^{\infty} \frac{1}{n^3} \rightarrow \text{Converges}$$

$$\sum_{n=1}^{\infty} \frac{(9-7)^n}{n^3(-2)^n} = \sum_{n=1}^{\infty} \frac{2^n}{n^3(-2)^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^3} \rightarrow \text{Converges}$$

$[5, 9]$

Geometry & Power series

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \ldots$$

$r = x \rightarrow$ Converges when $|r| < 1$

- When $-1 < x < 1$ it converges to $\frac{1}{1-x}$

Ex 1: $\frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{n=0}^{\infty} (-x)^n = \sum_{n=0}^{\infty} (-1)^n x^n$

Ex 2: $\frac{1}{1-11x} = \sum_{n=0}^{\infty} (11x)^n$

Ex 3: $\frac{1}{3+x} = \frac{1}{1-(-x-2)} = \sum_{n=0}^{\infty} (-x-2)^n = \sum_{n=0}^{\infty} (-1)^n (x+2)^n$

or

$$\frac{1}{3-x} = \frac{1}{3\left(1-\frac{x}{3}\right)} = \frac{1}{3} \sum_{n=0}^{\infty} \left(\frac{x}{3}\right)^n$$

Integrating and deriving power series

$$f(x) = \sum_{n=0}^{\infty} C_n x^n = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + \dots$$

$$f'(x) = \sum_{n=0}^{\infty} C_n \cdot n x^{n-1} = C_1 + 2C_2 x + 3C_3 x^2 + 4C_4 x^3 + \dots$$

$$\int f(x)\,dx = \sum_{n=0}^{\infty} \frac{C_n x^{n+1}}{n+1} = C_0 x + \frac{C_1}{2}x^2 + \frac{C_2}{3}x^3 + \frac{C_3}{4}x^4 + \dots$$

☆ They all have the same R

Ex: $f(x) = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ when $x: (-1,1)$

$$f'(x) = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} = \int \frac{1}{1-x}\,dx = -\ln(1-x) + C = C + x + \frac{x^2}{2} + \frac{x^3}{3} + \dots$$

To find C: $f'(0) = -\ln(1-0) = 0 = C + 0 + \frac{0^2}{2} + \frac{0^3}{3} + \dots$

$C = 0$

$$-\ln(1-x) = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}$$

Maclaurin series

- Find a power series which matches $f(x)$ on some interval centered at $x=0$.

- Assume the nth derivative exists for $f(0)$ for every n.
    - $f^{(n)}(0)$ exists

$$f(x) = C_0 + C_1x + C_2x^2 + C_3x^3 + C_4x^4 + C_5x^5 + \dots$$
$$f(0) = C_0$$

$$f'(x) = C_1 + 2C_2x + 3C_3x^2 + 4C_4x^3 + 5C_5x^4 + \dots$$
$$f'(0) = C_1$$

$$f''(x) = 2C_2 + 6C_3x + 12C_4x^2 + 20C_5x^3 + \dots$$
$$f''(0) = 2C_2$$

$$f'''(x) = 6C_3 + 24C_4x + 60C_5x^2 + \dots$$
$$f'''(0) = 6C_3$$

$$f^{(4)}(x) = 24C_4 + 120C_5x + \dots$$
$$f^{(4)}(0) = 24C_4$$

$$f^{(5)}(x) = 120C_5 + \dots$$
$$f^{(5)}(0) = 120C_5$$

$$C_n = \frac{f^{(n)}(0)}{n!}$$

$$\boxed{\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n}$$

Ex: Find maclaurin series for $f(x) = e^x$

$f'(x) = e^x, \ f''(x) = e^x, \ \ldots$

$f'(0) = 1, \ f''(0) = 1, \ \ldots$

$$\sum_{n=0}^{\infty} \frac{1}{n!} x^n$$

Ex 2: Find maclaurin series for $f(x) = \sin x$

$f(x) = \sin x \qquad f(0) = 0$

$f'(x) = \cos x \qquad f'(0) = 1$

$f''(x) = -\sin x \qquad f''(0) = 0$

$f'''(x) = -\cos x \qquad f'''(0) = -1$

$f^{(4)}(x) = \sin x \qquad f^{(4)}(0) = 0$

$$f(x) = 0 + \frac{1}{1!}x^1 + 0 + \frac{-1}{3!}x^3 + 0 + \frac{1}{5!}x^5 + \ldots$$

$$= x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \frac{1}{9!}x^9 + \ldots$$

$n = 0 \quad 1 \quad 2 \quad 3 \quad 4$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$$

$$\cos x = \frac{d}{dx} \sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \cdot 2n+1 \ x^{2n} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$$

Taylor series: Maclaurin series centered at point $x = a$

$$\boxed{\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n}$$

Ex:

$f(x) = x^4 - 3x^3 + 5x^2 + 2x - 8$

Find the taylor Series centered at $x=1$

$f(1) = 1 - 3 + 5 + 2 - 8$
$= -2 + 7 - 8 = 5 - 8 = \underline{-3}$

$f'(x) = 4x^3 - 9x^2 + 10x + 2$

$f'(1) = 4 - 9 + 10 + 2 = -5 + 12 = \underline{7}$

$f''(x) = 12x^2 - 18x + 10$

$f''(1) = 12 - 18 + 10 = -6 + 10 = \underline{4}$

$f'''(x) = 24x - 18$

$f'''(1) = 24 - 18 = \underline{6}$

$f^{(4)}(x) = \underline{24}$

$f^{(5)}(x) = \underline{0}$

$-3, 7, 4, 6, 24, 0, 0, 0$

Taylor series:
$-3 + 7(x-1) + 2(x-1)^2 + (x-1)^3 + (x-1)^4$
Just re-centers $f(x)$.