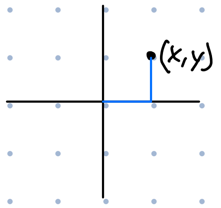
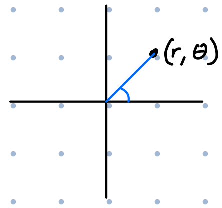
Cartesian	Polar	Polar → Cartesian	Cartesian → Polar





Polar → Cartesian

$x = r\cos\theta$

$y = r\sin\theta$

Cartesian → Polar

$r^2 = x^2 + y^2$

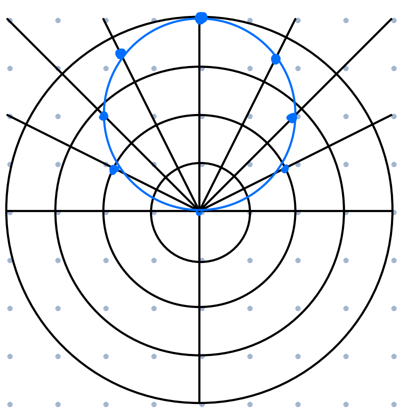
$\tan\theta = \frac{y}{x}$

- Double check it's in the correct quadrant.

Graphing Polar Curves

Ex 1: $r = 4\sin\theta$

θ	r
0	0
$\frac{\pi}{6}$	2
$\frac{\pi}{4}$	$2\sqrt{2} \approx 2.8$
$\frac{\pi}{3}$	$2\sqrt{3} \approx 3.4$
$\frac{\pi}{2}$	4
π	0
$\frac{7\pi}{6}$	-2



convert to cartesian coordinate:

$r = 4\sin\theta \rightarrow r^2 = 4r\sin\theta$

$\rightarrow x^2 + y^2 = 4y \rightarrow x^2 + y^2 - 4y = 0$

$\rightarrow x^2 + y^2 - 4y + 4 = 4$

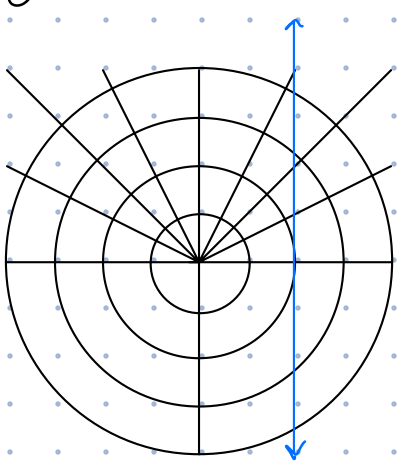
$\rightarrow x^2 + (y-2)^2 = 4$

Ex 2: $r = 2\sec\theta$

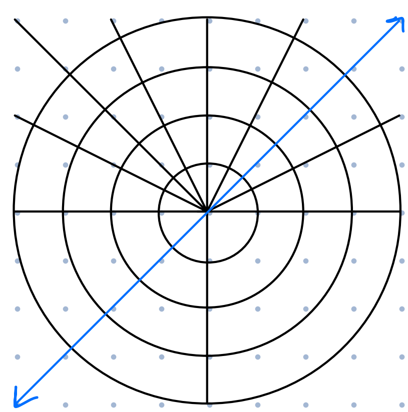
$r = \frac{2}{\cos\theta}$

$r\cos\theta = 2$

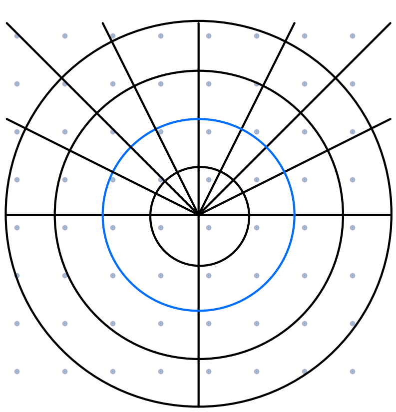
$x = 2$



Ex 3: $\theta = \frac{\pi}{4}$



Ex 4: $r = 2$



- It can be helpful plotting r in terms of θ

Symetry

About x-axis: $r(\theta) = r(-\theta)$

Rotate 180°: $r(\theta) = -r(\theta)$ or $r(\theta) = r(\theta + \pi)$

About y-axis: $r(\theta) = r(\pi - \theta)$

Slope of tangent line

$r = f(\theta)$

$$x = r\cos\theta = f(\theta)\cos\theta \qquad y = r\sin\theta = f(\theta)\sin\theta \xrightarrow{\text{Product rule}} \frac{dx}{d\theta} = f'(\theta)\cos\theta - f(\theta)\sin\theta \qquad \frac{dy}{d\theta} = f'(\theta)\sin\theta + f(\theta)\cos\theta$$

$$\frac{dy}{dx} = \frac{dy}{d\theta}\cdot\frac{d\theta}{dx} = \frac{dy/d\theta}{dx/d\theta}$$

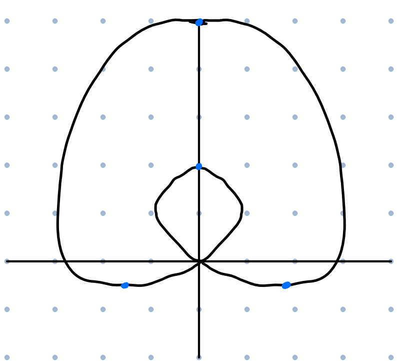
$$\boxed{\frac{dy}{dx} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}}$$

Ex 1: Slope at 1st time $r=\theta$ passes the positive y-axis.

$$\frac{dr}{d\theta} = 1 \qquad \frac{dy}{dx} = \frac{\sin\theta + \theta\cos\theta}{\cos\theta - \theta\sin\theta} \qquad @\ \theta = \frac{\pi}{2}$$

$$\frac{dy}{dx}\left(\frac{\pi}{2}\right) = \frac{\sin\frac{\pi}{2} + \frac{\pi}{2}\cos\frac{\pi}{2}}{\cos\frac{\pi}{2} - \frac{\pi}{2}\sin\frac{\pi}{2}} = \frac{1+0}{0-\frac{\pi}{2}} = -\frac{2}{\pi}$$

Ex 2: Find all points where there's a horizontal tangent line for $r = 1 + 2\sin\theta$



$$\frac{dr}{d\theta} = 2\cos\theta$$

$$\frac{dx}{dy} = \frac{2\cos\theta\sin\theta + (1+2\sin\theta)\cos\theta}{2\cos^2\theta - (1+2\sin\theta)\sin\theta}$$

$\frac{dx}{dy} = 0$ for horizontal tangent lines so $0 =$ numerator

$$0 = 2\cos\theta\sin\theta + (1+2\sin\theta)\cos\theta$$
$$0 = \cos\theta(2\sin\theta + 1 + 2\sin\theta)$$
$$0 = \cos\theta(1 + 4\sin\theta)$$

$0 = \cos\theta$

$\theta = \frac{\pi}{2}$ or $\theta = \frac{3\pi}{2}$

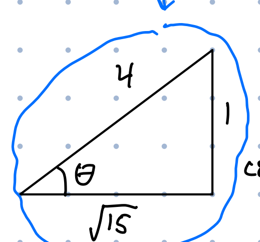
$r = 3$	$r = -1$
$x = 3\cos\frac{\pi}{2} = 0$	$x = -1\cdot\cos\frac{3\pi}{2} = 0$
$y = 3\sin\frac{\pi}{2} = 3$	$y = -1\cdot\sin\frac{3\pi}{2} = -1\cdot -1 = 1$
$(0, 3)$	$(0, 1)$

$0 = 1 + 4\sin\theta \rightarrow -\frac{1}{4} = \sin\theta$

$\theta = \sin^{-1}\left(-\frac{1}{4}\right)$

$r = 1 + 2\left(-\frac{1}{4}\right) = 1 - \frac{1}{2} = \frac{1}{2}$

$$x = \frac{1}{2}\cos\left(\sin^{-1}\left(-\frac{1}{4}\right)\right) = \frac{1}{2}\cdot\pm\frac{\sqrt{15}}{4}$$



$4^2 = 1^2 + x^2$

$x = \sqrt{15}$

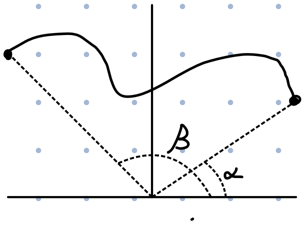
$-+$ | $++$

$\cos\theta = \frac{adj}{hyp} = \frac{\sqrt{15}}{4}$

$$y = \frac{1}{2}\sin\left(\sin^{-1}\left(-\frac{1}{4}\right)\right) = \frac{1}{2}\cdot -\frac{1}{4} = -\frac{1}{8}$$

$$\left(\pm\frac{\sqrt{15}}{8}, -\frac{1}{8}\right)$$

Arc Length



$$x = r\cos\theta = f(\theta)\cos\theta$$
$$y = r\sin\theta = f(\theta)\sin\theta$$

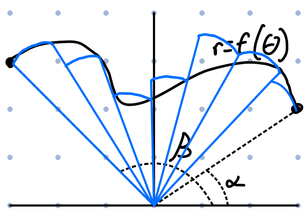
$$\frac{dx}{d\theta} = \frac{dr}{d\theta}\cos\theta - r\sin\theta$$
$$\frac{dy}{d\theta} = \frac{dr}{d\theta}\sin\theta + r\cos\theta$$

$$L = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dr}{d\theta}\cos\theta - r\sin\theta\right)^2 + \left(\frac{dr}{d\theta}\sin\theta + r\cos\theta\right)^2}\, d\theta$$

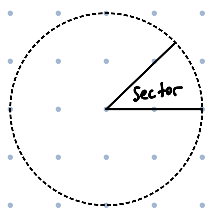
$$= \int_{\alpha}^{\beta} \sqrt{\begin{matrix} \left(\frac{dr}{d\theta}\right)^2\cos^2\theta - \cancel{2r\frac{dr}{d\theta}\cos\theta\sin\theta} + r^2\sin^2\theta + \\ \underbrace{\left(\frac{dr}{d\theta}\right)^2\sin^2\theta}_{\left(\frac{dr}{d\theta}\right)^2} + \cancel{2r\frac{dr}{d\theta}\cos\theta\sin\theta} + \underbrace{r^2\cos^2\theta}_{r^2} \end{matrix}}\, d\theta$$

$$\boxed{L = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dr}{d\theta}\right)^2 + r^2}\, d\theta}$$

Area under a curve



Divide into ∞ many sectors of a circle with $\theta = d\theta$.



Area of sector $= \frac{\theta}{2\pi} \cdot \pi r^2 = \frac{1}{2} r^2 \theta$

In radians

Area of i^{th} sector: $\frac{1}{2} f(\theta_i)^2 \Delta\theta$

$$\text{Area} \approx \sum_{i=1}^{n} \frac{1}{2} f(\theta_i)^2 \Delta\theta$$

$$\boxed{\text{Area} = \frac{1}{2}\int_{\alpha}^{\beta} r^2\, d\theta}$$