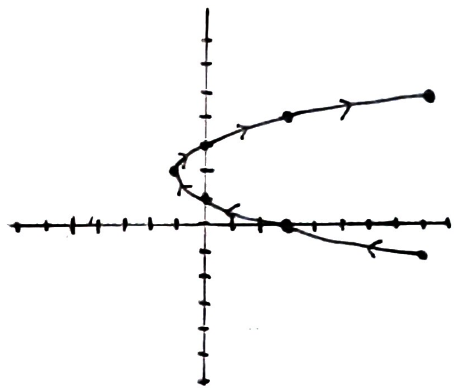
Parametric Equations

Parametric equations express x and y in terms of a 3^{rd} variable.

Ex: Graph $x = t^2 - 2t$ and $y = t + 1$

t	x	y
-2	8	-1
-1	3	0
0	0	1
1	-1	2
2	0	3
3	3	4
4	8	5



Sketch the curve with increasing values of t.

Finding a Cartesian parameter:

1. Use one of the equations to solve for t
2. Plug in t into the other equation

$y = t + 1 \Rightarrow t = y - 1$

$x = t^2 - 2t \Rightarrow x = (y-1)^2 - 2(y-1) \Rightarrow x = y^2 - 4y + 3$

Finding derivatives:

chain rule: $\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$

$$\boxed{\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}} \qquad \frac{d}{dx}\left(\frac{dy}{dx}\right) = \boxed{\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}}$$

$$\frac{dy}{dt} = 1 \qquad \frac{dx}{dt} = 2t - 2 \qquad \frac{d}{dt}\left(\frac{dy}{dx}\right) = -\frac{1}{(2t-2)^2}(2)$$

$$\frac{dy}{dx} = \frac{1}{2t-2} \qquad \frac{d^2y}{dx^2} = \frac{-\frac{2}{(2t-2)^2}}{2t-2} = -\frac{2}{(2t-2)^3}$$

Area under a curve:

$$A = \int_a^b y\,dx \qquad y = g(t) \qquad x = f(t)$$
$$dx = f'(t)\,dt = \frac{dx}{dt}\,dt$$

Since we're changing the variable from x to t, we need to change the bounds.
Lower: $a = f(t)$ solve for t which we'll call α.
Upper: $b = f(t)$ solve for t which we'll call β

$$= \int_\alpha^\beta g(t)\frac{dx}{dt}\,dt$$

Arc length:

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx \qquad y = g(t) \qquad x = f(t)$$
$$dx = f'(t)\,dt = \frac{dx}{dt}\,dt$$
$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

$$= \int_\alpha^\beta \sqrt{1 + \left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right)^2}\,\frac{dx}{dt}\,dt = \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

Ex: Find arc length of $x = \cos t$, $y = t + \sin t$ from $0 \le t \le \pi$

$$\frac{dx}{dt} = -\sin t \qquad \frac{dy}{dt} = 1 + \cos t$$

$$L = \int_0^\pi \sqrt{(-\sin t)^2 + (1+\cos t)^2}\,dt = \int_0^\pi \sqrt{1 + 2\cos t + \cos^2 t + \sin^2 t}\,dt$$

$$(1+\cos t)^2 = 1 + 2\cos t + \cos^2 t \qquad \cos^2 t + \sin^2 t = 1$$

$$= \int_0^\pi \sqrt{1 + 2\cos t + 1}\,dt = \int_0^\pi \sqrt{2\cos t + 2}\,dt = \int_0^\pi \sqrt{2(\cos t + 1)}\,dt$$

$$\cos^2 t = \frac{1}{2}(1 + \cos 2t) \Rightarrow \cos^2\left(\frac{t}{2}\right) = \frac{1}{2}(1 + \cos t) \Rightarrow 2\cos^2\left(\frac{t}{2}\right) = 1 + \cos t$$

$$= \int_0^{\pi} \sqrt{2\left(2\cos^2\left(\frac{t}{2}\right)\right)}\, dt = \int_0^{\pi} \sqrt{2^2\cos^2\left(\frac{t}{2}\right)}\, dt = \int_0^{\pi} 2\cos\left(\frac{t}{2}\right) dt$$

$$= 2\left[2\sin\left(\frac{t}{2}\right)\right]_0^{\pi} = 4\left(\sin\frac{\pi}{2} - \sin 0\right) = 4(1-0) = 4$$

<u>Surface area when revolved around an axis:</u>

$$SA = \int_a^b 2\pi y\sqrt{1 + (f'(x))^2}\, dx = \int_\alpha^\beta 2\pi g(t)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$