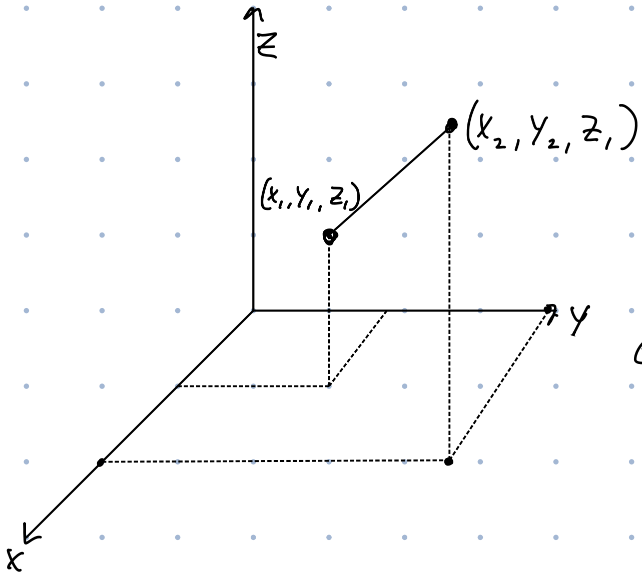
3D distance formula

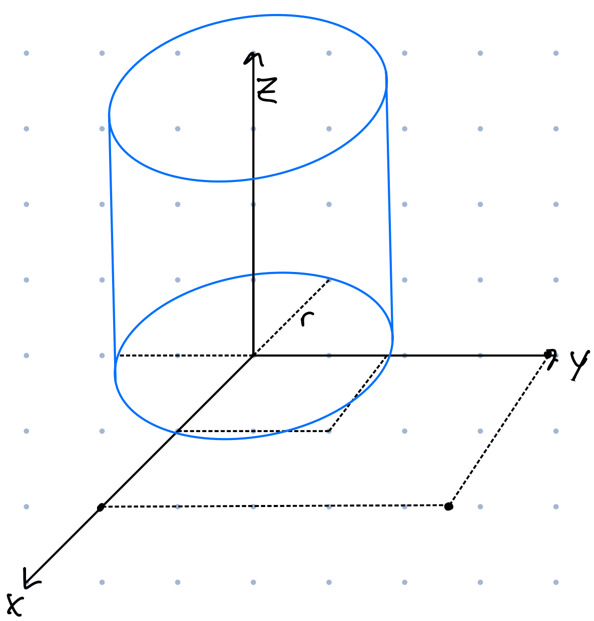


3D Distance formula $= \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$

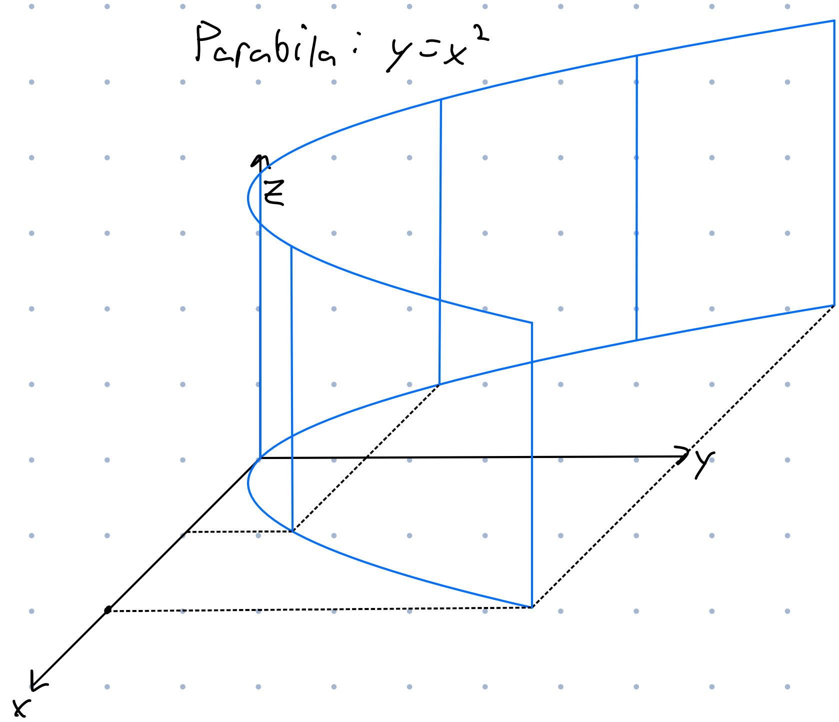
The coordinate planes (xy, yz, and xz) cut space into 8 octants.

1^{st} octant: $x > 0, y > 0, z > 0$

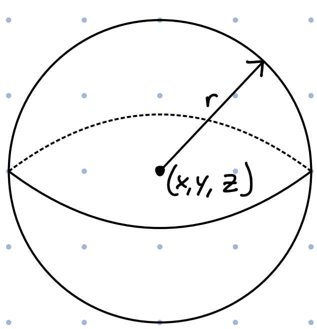
Cylinders: $x^2 + y^2 = r^2$



Parabila: $y = x^2$



Spheres: $r^2 = (x-a)^2 + (y-b)^2 + (z-c)^2$

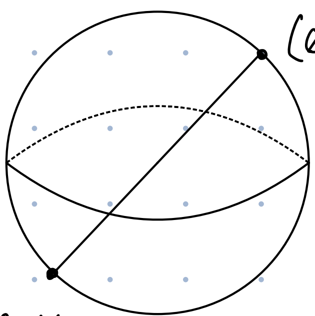


Ex 1: Is $x^2 - 4x + y^2 + 6y + z^2 - 12z = 15$ a sphere?

$$(x^2 - 4x + 4) + (y^2 + 6y + 9) + (z^2 - 12 + 36) = 15 + 4 + 9 + 36$$

$$(x-2)^2 + (y+3)^2 + (z-6)^2 = 8^2 \qquad \text{Yes}$$

Ex 2: What is the equation for the sphere who's diameter has those points?



$$r = \frac{\sqrt{(0-12)^2 + (6-4)^2 + (7-23)^2}}{2} = \sqrt{101}$$

$$\text{center} = \left(\frac{0+12}{2}, \frac{6+4}{2}, \frac{7+23}{2}\right) = (6, 5, 15)$$

$$101 = \sqrt{(x-6)^2 + (y-5)^2 + (z-15)^2}$$

Lines in 3D

Lines in 3D are defined by a vector and a point.

Ex: Find the equation of a line that goes through the points $(1,2,3)$ and $(3,3,0)$

Direction vector = Slope of a 3D line $= \langle 3-1, 3-2, 0-3 \rangle = \langle 2, 1, -3 \rangle$

Vector equation

Point, Direction vector

$$r = \underbrace{(1,2,3)}_{\text{Point}} + t\underbrace{(2,1,-3)}_{\text{Direction vector}}$$

or

$$\vec{r}(t) = \langle 1+2t,\ 2+t,\ 3-3t \rangle$$

Parametric equation

Point, Direction vector

$$x = 1 + 2t$$
$$y = 2 + t$$
$$z = 3 - 3t$$

Symmetric equation

Solve for t

$$\frac{x-1}{2} = y-2 = \frac{z-3}{-3}$$

At what point does the line intersect the plane $3x+y=3$?

1) Solve for t

$$3(1+2t) + (2+t) = 3 \rightarrow 3+6t+2+t=3 \rightarrow 5+7t=3 \rightarrow 7t=-2$$

$$t = -\frac{2}{7}$$

2) Find point

$$x = 1 + 2\left(-\frac{2}{7}\right) = \frac{3}{7}$$
$$y = 2 + \left(-\frac{2}{7}\right) = \frac{12}{7}$$
$$z = 3 - 3\left(-\frac{2}{7}\right) = \frac{27}{7}$$

$$\left(\frac{3}{7}, \frac{12}{7}, \frac{27}{7}\right)$$

At what point does it intersect the line $r = (0, 10, \frac{14}{5}) + t(1,-2,-1)$?

1) Convert line to a different variable(s).

$x = s$

$y = 10 - 2s$

$z = \frac{14}{5} - s$

2) Set x, y, and z equal and solve for both variables.

$s = 1 + 2t$

$10 - 2s = 2 + t \rightarrow 10 - 2(1 + 2t) = 2 + t \rightarrow 10 - 2 - 4t = 2 + t$

$\rightarrow 8 - 4t = 2 + t \rightarrow 8 = 2 + 5t \rightarrow 6 = 5t$

$t = \frac{6}{5}$

$s = 1 + 2\left(\frac{6}{5}\right) = 1 + \frac{12}{5} = \frac{5}{5} + \frac{12}{5}$

$s = \frac{17}{5}$

3) If both variables solve all 3 equations, then the lines intersect.

$\frac{14}{5} - s = 3 - 3t$

$\frac{14}{5} - \left(\frac{17}{5}\right) = 3 - 3\left(\frac{6}{5}\right) \rightarrow -\frac{3}{5} = 3 - \frac{18}{5} \rightarrow -\frac{3}{5} = \frac{15}{5} - \frac{18}{5} = -\frac{3}{5}$ ✓

4) Find the point

$x = \left(\frac{17}{5}\right)$

$y = 10 - 2\left(\frac{17}{5}\right) = \frac{50}{5} - \frac{34}{5} = \frac{16}{5}$

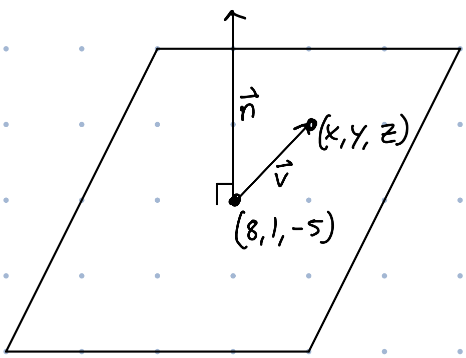
$z = \frac{14}{5} - \left(\frac{17}{5}\right) = -\frac{3}{5}$

$\left(\frac{17}{5}, \frac{16}{5}, -\frac{3}{5}\right)$

Planes in 3D

Planes in 3D are defined by a normal vector and point.

Ex: Equation for a plane with normal vector $\langle 2,3,4\rangle$ which contains the point $(8,1,-5)$.



$\vec{v} = \langle x-8, y-1, z+5\rangle$

$\vec{n}\cdot\vec{v} = 0$ because they have to be $\perp$.

$$\vec{n}\cdot\vec{v} = (2)(x-8) + (3)(y-1) + (4)(z+5) = 0$$
$$= 2x - 16 + 3y - 3 + 4z + 20 = 0$$
$$= 2x + 3y + 4z + 1 = 0$$
$$\underbrace{2x + 3y + 4z}_{\vec{n}} = -1$$

You could also do:

$$n_x x + n_y y + n_z z = D$$

$$2x + 3y + 4z = D \rightarrow 2(8) + 3(1) + 4(-5) = -1$$

Find the plane from 3 points: $(1,0,2)$, $(3,1,2)$, and $(1,2,4)$

1) Create 2 vectors on the plane.

$$\vec{v} = \langle 3-1, 1-0, 2-2\rangle = \langle 2,1,0\rangle$$
$$\vec{w} = \langle 1-1, 2-0, 4-2\rangle = \langle 0,2,2\rangle$$

2) $\vec{v} \times \vec{w} = \vec{n}$

Determinant

$$\vec{v} \times \vec{w} = \left[\begin{array}{ccc|cc} \hat{i} & \hat{j} & \hat{k} & \hat{i} & \hat{j} \\ 2 & 1 & 0 & 2 & 1 \\ 0 & 2 & 2 & 0 & 2 \end{array}\right] \rightarrow \begin{array}{l} (1)(2)\hat{i} + (0)(0)\hat{j} + (2)(2)\hat{k} - \\ (0)(2)\hat{i} - (2)(2)\hat{j} - (1)(0)\hat{k} \\ = 2\hat{i} - 4\hat{j} + 4\hat{k} \end{array}$$

3) Equation for a plane

$2x - 4y + 4z = D$

$2(1) - 4(0) + 4(2) = 2 + 8 = 10$

$2x - 4y + 4z = 10$

Find the line from these intersecting planes:

Plane 1: $3x - 2y + z = 1$

Plane 2: $2x + y - 3z = 3$

1) Set one variable to t.

$\underline{x = t}$

$3t - 2y + z = 1$

$2t + y - 3z = 3$

2) Solve for the other two variables in terms of t.

$3t - 2y + z = 1 \rightarrow z = 1 + 2y - 3t$

$2t + y - 3(1 + 2y - 3t) = 3$

$2t + y - 3 - 6y + 9t = 3$

$-5y + 11t - 3 = 3 \rightarrow -5y = 6 - 11t$

$\underline{y = -\frac{6}{5} + \frac{11}{5}t}$

$z = 1 + 2\left(-\frac{6}{5} + \frac{11}{5}t\right) - 3t = 1 - \frac{12}{5} + \frac{22}{5}t - 3t$

$\underline{z = -\frac{7}{5} + \frac{7}{5}t}$

Find the angle between those planes

1) Find normal vectors of planes.

$\vec{n}_1 = \langle 3, -2, 1 \rangle$

$\vec{n}_2 = \langle 2, 1, -3 \rangle$

2) Find angle between normal vectors

$\vec{n}_1 \cdot \vec{n}_2 = n_1 n_2 \cos\theta$

$\vec{n}_1 \cdot \vec{n}_2 = (3)(2) + (-2)(1) + (1)(-3) = 6 - 2 - 3 = 1$

$n_1 = \sqrt{3^2 + (-2)^2 + 1^2} = \sqrt{9 + 4 + 1} = \sqrt{14}$

$n_2 = \sqrt{2^2 + 1^2 + (-3)^2} = \sqrt{4 + 1 + 9} = \sqrt{14}$

$\theta = \cos^{-1}\left(\frac{1}{\sqrt{14}\sqrt{14}}\right) = 85.9^\circ$

Other problems

- Distance from point to a plane
  1) Find $\vec{n}$
  2) Create line w/ $\vec{n}$ and point
  3) Find intersection point
  4) Distance formula
- Plane from line and perpendicular to plane.
  1) $\vec{n}$ of plane
  2) Direction vector of line ($\vec{v}$)
  3) $\vec{v} \times \vec{n} = \vec{n}$ of new plane
  4) Poin on line to find D