Use cases for integrals:

Net change theorem

$$\int_a^b f'(x)\,dx = f(b) - f(a)$$

The integral of the rate of change ($f'(x)$) is the net change ($f(b) - f(a)$).

Ex: Suppose you have a bird's eye view of a car and know it's x and y velocity in meters per sec.

$V_x(t) = 1$ $\qquad$ $V_y(t) = 1 + \cos(2t)$

What is its displacement and distance traveled after 5 sec?

$$\Delta x = \int_0^5 1\,dt = t\Big|_0^5 = 5$$

$$\Delta y = \int_0^5 1 + \cos(2t)\,dt = t + \frac{1}{2}\sin(2t)\Big|_0^5 = 5 + \frac{1}{2}\sin(10)$$

Displacement from origin: $\sqrt{(5)^2 + \left(5 + \frac{1}{2}\sin(10)\right)^2} = 6.88$ meters

Total distance traveled:

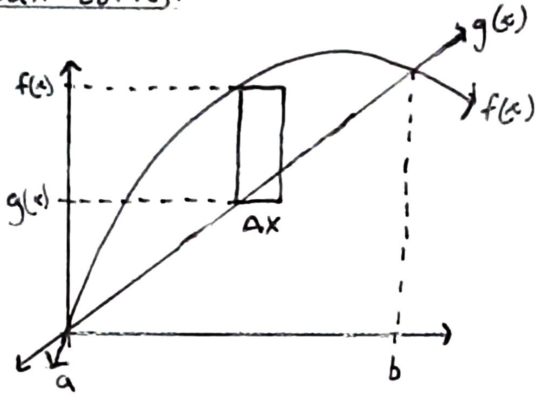
$$\text{Speed}(t) = \sqrt{V_x^2 + V_y^2}$$

$$\int_0^5 \text{Speed}(t)\,dt = \int_0^5 \sqrt{25 + (1 + \cos(2t))^2}\,dt \approx 7.42 \text{ meters}$$

Total distance traveled on the x $= \int_0^5 |V_x|\,dt = 5$ meters

Total distance traveled on the y $= \int_0^5 |V_y|\,dt = 4.73$ meters

Area between curves:



$$\text{Area} = \int_a^b f(x) - g(x)\,dx$$

$$= \left| \int_a^b g(x) - f(x)\,dx \right|$$

1. Find a and b usually by finding intersection points.
2. Find which function is ontop or use absolute value.
3. Solve definite integral

Ex: Find area bounded by $y = \sin x$ and $y = \cos x$ on $[0, \frac{\pi}{2}]$

$\sin x = \cos x \qquad x = \frac{\pi}{4}$

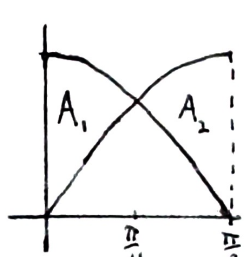
$$\text{Area} = \left| \int_0^{\frac{\pi}{4}} \sin x - \cos x\,dx \right| + \left| \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin x - \cos x\,dx \right|$$

$$= \left| -\cos x - \sin x \Big|_0^{\frac{\pi}{4}} \right| + \left| -\cos x - \sin x \Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}} \right|$$

$$= \left| -\cos\frac{\pi}{4} - \sin\frac{\pi}{4} - \left(-\cos 0 - \sin 0\right) \right| + \left| -\cos\frac{\pi}{2} - \sin\frac{\pi}{2} - \left(-\cos\frac{\pi}{4} - \sin\frac{\pi}{4}\right) \right|$$

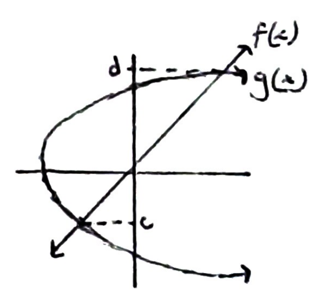
$$= \left| -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} - (-1 - 0) \right| + \left| -0 - 1 - \left(-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\right) \right| = \left| -\sqrt{2} + 1 \right| + \left| -1 + \sqrt{2} \right|$$

$$= \sqrt{2} - 1 + \sqrt{2} - 1 = 2(\sqrt{2} - 1) = 2\sqrt{2} - 2$$



You can also integrate with respect to y

$$\int_c^d f(x) - g(x)\,dx$$



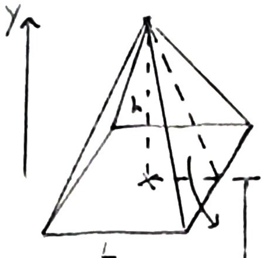
Volumes with cylinders

Volume of a cylinder = Area of base · Height
(Even for irregular base)

The volume of objects other than cylinders can be found by cutting it into infinitly many cylinders where the area of its base is $A(x)$ or $A(y)$ and its height is dx or dy.

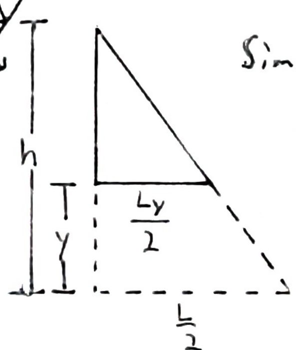
$$\int_a^b A(x)\,dx \quad \text{or} \quad \int_c^d A(y)\,dy$$

Ex 1: Find the volume of a pyramid with a square base of length L and height of h.



$$\int_0^h A(y)\,dy = \int_0^h L_y^{\,2}\,dy$$

Cylinders stack in the y direction so it's dy.



Similar triangles:

$$\frac{h}{\frac{L}{2}} = \frac{h-y}{\frac{L_y}{2}} \Rightarrow \frac{2h}{L} = \frac{2(h-y)}{L_y}$$

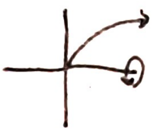
$$L_y = \frac{L}{h}(h-y) = L - \frac{L}{h}y$$

$$\int_0^h \left(L - \frac{L}{h}y\right)^2 dy = \int_0^h L^2 - 2\frac{L^2}{h}y + \frac{L^2}{h^2}y^2\,dy = L^2\int_0^h 1 - \frac{2}{h}y + \frac{1}{h^2}y^2\,dy$$

$$= L^2\left[y - \frac{1}{h}y^2 + \frac{1}{3h^2}y^3\right]_0^h = L^2\left(h - \frac{h^2}{h} + \frac{h^3}{3h^2}\right) = \frac{1}{3}L^2h$$

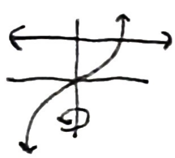
When an area is revolved around an axis, the area of the base is a circle (πr^2) and you need to write r in terms of x or y.

Ex 2: Find the volume of the solid obtained by rotating $y=\sqrt{x}$ around the x-axis from 0 to 1.



$$\int_0^1 \pi(\sqrt{x})^2\,dx = \pi\int_0^1 x\,dx = \pi\left[\frac{1}{2}x^2\right]_0^1 = \frac{\pi}{2}$$

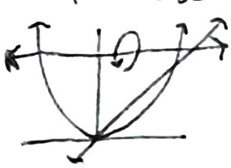
Ex 3: Find the volume of the solid obtained by rotating the area bounded by $y=x^3$, $y=8$, and $x=0$ around the y-axis.



$$\int_0^8 \pi\left(y^{\frac{1}{3}}\right)^2 dy = \pi\int_0^8 y^{\frac{2}{3}}\,dy = \pi\left[\frac{3}{5}y^{\frac{5}{3}}\right]_0^8 = \pi\left(\frac{3}{5}8^{\frac{5}{3}}\right)$$

$$= \pi\frac{3}{5}\left(8^{\frac{1}{3}}\right)^5 = \pi\frac{3}{5}2^5 = \pi\frac{3\cdot 32}{5} = \frac{96}{5}\pi$$

Ex 4: Find the volume of the solid obtained by rotating the area enclosed by $y=x$ and $y=x^2$ around $y=2$



Find the bounds: $x=x^2 \Rightarrow 0=x^2-x=x(x-1)$

$x=0$ and 1

$$\int_0^1 \pi\underbrace{(2-x^2)}_{\text{Outer radius}}{}^2 - \pi\underbrace{(2-x)}_{\text{Inner radius}}{}^2\,dx = \pi\int_0^1 4-4x^2+x^4-(4-4x+x^2)\,dx$$

$(2-x^2)(2-x^2)=4-4x^2+x^4$ $\quad$ $(2-x)(2-x)=4-4x+x^2$

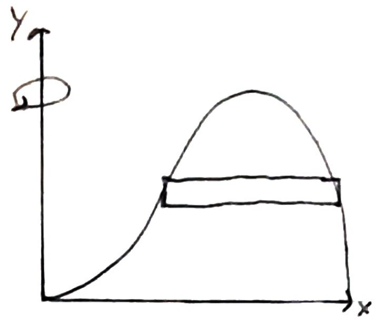
$$=\pi\int_0^1 x^4-5x^2+4x\,dx = \pi\left[\frac{1}{5}x^5-\frac{5}{3}x^3+\frac{4}{2}x^2\right]_0^1 = \pi\left(\frac{1}{5}-\frac{5}{3}+2\right)$$

$$=\pi\left(\frac{6}{30}-\frac{50}{30}+\frac{60}{30}\right)=\pi\left(-\frac{44}{30}+\frac{60}{30}\right)=\frac{16}{30}\pi=\frac{8}{15}\pi$$

Volume with shells

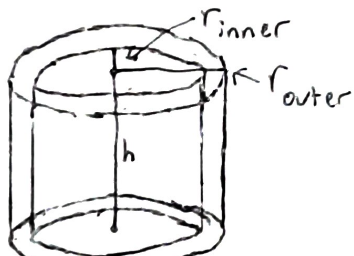
It can be hard to stack cylinders for some objects.

Ex: Find the volume obtained by rotating $y=2x^2-x^3$ and $y=0$ around the y-axis



If we were to use cylinders, we would have to find the local max and then divide $y=2x^2-x^3$ into two separate functions for the outer and inner radius of the circle.

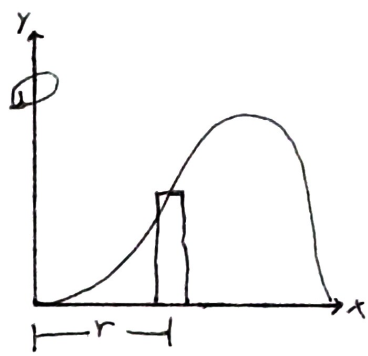
Instead we could use an infinite number of shells.



$$\text{Volume} = \text{Volume}_{outer} - \text{Volume}_{inner}$$
$$= \pi r_{out}^2 h - \pi r_{in}^2 h$$
$$= \pi (r_{out}^2 - r_{in}^2) h$$
$$= \pi (r_{out} + r_{in})(r_{out} - r_{in}) h$$
$$= \pi (r_{out} + r_{in}) \Delta r\, h$$
$$= 2\pi \frac{r_{out} + r_{in}}{2} h \,\Delta r$$
$$= \underbrace{2\pi r_{ave}}_{\text{Circumference}} \cdot \underset{\text{height}}{h} \cdot \underbrace{\Delta r}_{\text{Thickness}}$$

$$\text{Volume} = \int_a^b 2\pi r \cdot f(r) \cdot dr$$

Ex:



$$\int_0^2 2\pi x \cdot (2x^2 - x^3)\, dx = \frac{16}{5}\pi$$

Use shells over cylinders when the axis of rotation (Ex: y-axis) is different from the independent variable (Ex: x). $y \neq x$

Work

$F = ma$ $\qquad W = F \cdot d$ when Force is constant

Springs: $F = kx$

$$W = \int_{x_i}^{x_f} F(x) \cdot dx = \int_{x_i}^{x_f} h(x) \cdot dF(x)$$ when Force isn't constant.

	Force	Distance	Work	g
Metric	N	m	J	9.8
Imperial	lbs	ft	ft-lbs	32.2

Ex 1: A force of 40N is required to hold a spring that has been stretched from its natural length of 10cm to a length of 15cm. How much work is done by stretching it from 15cm to 18cm?

$F = kx \qquad 40 = k(15-10) \qquad k = 8\ \frac{N}{cm}$

$$W = \int_5^8 8x\, dx = 4x^2\Big|_5^8 = 4 \cdot 8^2 - 4 \cdot 5^2 = 156\ Ncm \cdot \frac{1m}{100cm} = 1.56\ J$$

Ex 2: A 200 lb cable is 100ft long and hangs vertically from the top of a tall building. How much work to lift the cable to the top?

x - How far up the cable was pulled.

$F(x)$ = Force of cable hanging

$F(0) = 200 \qquad F(100) = 0$

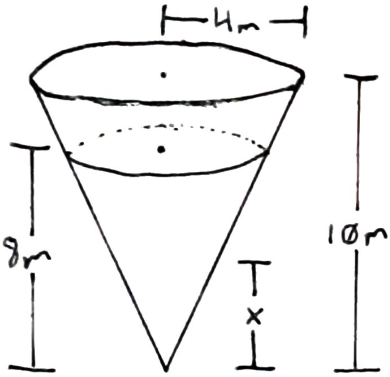
Assuming uniform density of the cable, the Force is linear/a line.

$$\text{slope} = \frac{F(0) - F(100)}{0 - 100} = \frac{200}{-100} = -2 \qquad F(x) - 0 = -2(x - 100)$$

$$W = \int_0^{100} -2(x-100)\, dx = -2\int_0^{100} x - 100\, dx = -2\left[\tfrac{1}{2}x^2 - 100x\right]_0^{100} = 10{,}000\ \text{ft-lbs}$$

Ex 3: A tank has the shape of an inverted circular cone with height 10m and base radius 4m. It is filled with water to a height of 8m. Find the work required to empty the tank by pumping all of the water to the top of the tank?

Density of water is $1{,}000 \frac{kg}{m^3}$



x = Height of water left in the tank.

$$W = \int_0^8 h(x)\, dF(x) = \int_0^8 (10 - x)\, dF(x)$$

$dF(x)$ = Force from a small cylindrical slice.

= mass · acc = $dm(x) \cdot 9.8$

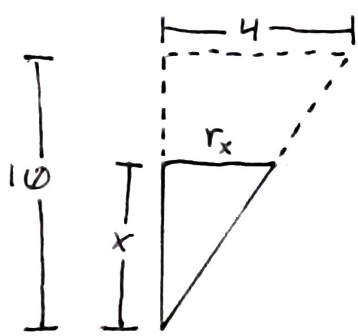
$dm(x)$ = Mass of small cylindrical slice.

= Density · Volume = $1000 \cdot dV(x)$

$dV(x)$ = Volume of small cylindrical slice.

= Area of base · height

= $\pi r_x^2 \cdot dx$



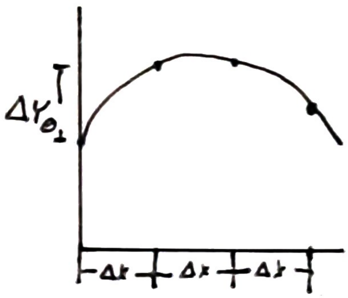
$$\frac{4}{10} = \frac{r_x}{x}$$

$$r_x = \frac{2}{5}x$$

$$W = \int_0^8 (10 - x) \cdot 9.8 \cdot 1{,}000 \cdot \pi \left(\frac{2}{5}x\right)^2 dx = 3.36 \times 10^6 \text{ J}$$

Arc Length

- Divide arc into infinity many line segments with equal width (Δx)
- Add all the line segment's distances up using the distance formula



$$\text{Distance of one segment} = \sqrt{(x_i - x_{i-1})^2 + (Y_i - Y_{i-1})}$$

$$= \sqrt{(\Delta x)^2 + (\Delta Y_i)^2}$$

$$\text{Slope} = \frac{\Delta Y_i}{\Delta x} = f'(x_i) \Rightarrow \Delta Y_i = f'(x_i)\,\Delta x$$

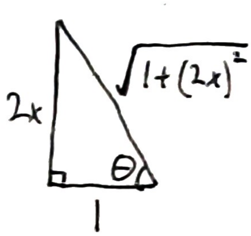
$$= \sqrt{(\Delta x)^2 + (f'(x_i)\,\Delta x)^2} = \sqrt{\Delta x^2\left(1 + (f'(x_i))^2\right)} = \sqrt{1 + (f'(x_i))^2}\,\Delta x$$

Sum them up $\Rightarrow$ $\underline{L = \int_a^b \sqrt{1 + (f'(x))^2}\,dx}$

Ex: Find the arc length of $y = x^2$ from 0 to 5

$$L = \int_0^5 \sqrt{1 + (2x)^2}\,dx \qquad \begin{aligned} 2x &= \tan\theta \\ 2dx &= \sec^2\theta\,d\theta \end{aligned} \qquad \int \sqrt{1 + \tan^2\theta}\,\frac{\sec^2\theta}{2}\,d\theta$$

$$= \frac{1}{2}\int \sec^3\theta\,d\theta = \frac{1}{4}\left(\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|\right) + C$$



$$\sec\theta = \frac{hyp}{adj} = \sqrt{1 + (2x)^2}$$

$$= \frac{1}{4}\left(\sqrt{1+(2x)^2}\,2x + \ln\left|\sqrt{1+(2x)^2} + 2x\right|\right)\Big|_0^5$$

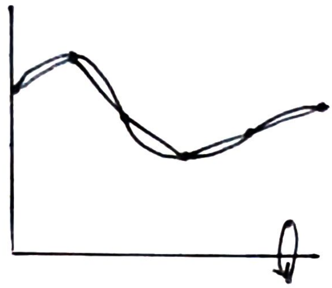
$$= 25.8742$$

You can also have a constant Δy and have ΔX_i

$$L = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\,dy$$

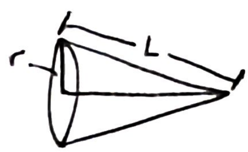
Surface area of an object of revolution

1. Approximate the function with line segments.



2. The surface area of a line segment is part of a circular cone

Surface area of a cone:



If you cut a cone along L and fold it out.

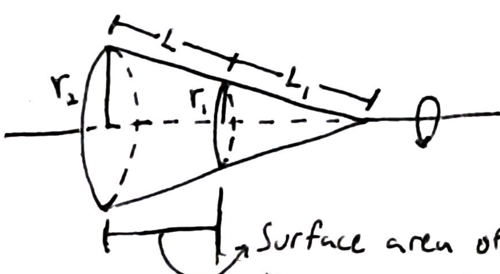
$$\text{Arc Length} = \frac{2\pi L}{\theta} = 2\pi r$$

$$\theta = \frac{2\pi L}{2\pi r} = \frac{L}{r}$$

$$S.A. = \frac{\pi L^2}{\theta}$$

$$= \pi L^2 \cdot \frac{r}{L} = \underline{\pi r L}$$

Surface area of a cone segment:



$$\text{S.A of larger cone} - \text{S.A of smaller cone}$$

Surface area of line segment $= \pi r_2 (L + L_1) - \pi r_1 L_1$

$$= \pi(r_2 L + r_2 L_1 - r_1 L_1) = \pi\left[L_1(r_2 - r_1) + r_2 L\right]$$

Similar triangles: $\dfrac{L+L_1}{r_2} = \dfrac{L_1}{r_1} \Rightarrow r_1(L+L_1) = L_1 r_2$

$$L r_1 + L_1 r_1 = L_1 r_2 \Rightarrow L_1 r_1 - L_1 r_2 = -L r_1 \Rightarrow L_1 r_2 - L_1 r_1 = L r_1$$

$$= L_1(r_2 - r_1) = L r_1$$

$\pi\left(L r_1 + L r_2\right) = 2\pi r L$ where $r = \dfrac{r_1 + r_2}{2}$ or the average of rs.

3. Add up infinitly many surface areas. A.K.A integrate

$$\int_a^b 2\pi r L\,dx = \int_a^b 2\pi f(x)\sqrt{1+(f'(x))^2}\,dx$$ for revolving around the x-axis.

$$= \int_c^d 2\pi y\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$$

or for revolving around the y-axis)

$$\int_a^b 2\pi x\sqrt{1+(f'(x))^2}\,dx \quad \text{or} \quad \int_c^d 2\pi g(y)\sqrt{1+(g'(y))^2}\,dy$$

Ex: Find the surface area of the function $y=\sqrt{x}$ revolved around the x-axis from (1,1) to (4,2)

$$S = \int_1^4 2\pi\sqrt{x}\sqrt{1+\left(\frac{1}{2\sqrt{x}}\right)^2}\,dx \quad \text{or} \quad S = \int_1^2 2\pi y\sqrt{1+(2y)^2}\,dy$$

$$= 2\pi\int_1^4 \sqrt{x}\sqrt{1+\frac{1}{4x}}\,dx = 2\pi\int_1^4 \sqrt{x}\sqrt{\frac{4x}{4x}+\frac{1}{4x}}\,dx = 2\pi\int_1^4 \sqrt{x}\sqrt{\frac{4x+1}{4x}}\,dx$$

$$= 2\pi\int_1^4 \sqrt{x}\,\frac{\sqrt{4x+1}}{\sqrt{4x}}\,dx = 2\pi\int_1^4 \sqrt{x}\,\frac{\sqrt{4x+1}}{2\sqrt{x}}\,dx = \pi\int_1^4 \sqrt{4x+1}\,dx$$

$u=4x+1 \quad du=4dx \Rightarrow dx=\frac{du}{4}$ Lower: $u=4(1)+1=5$

Upper: $u=4(4)+1=17$

$$= \pi\int_5^{17} \sqrt{u}\,\frac{du}{4} = \frac{\pi}{4}\left[\frac{2}{3}u^{3/2}\right]_5^{17} = \frac{\pi}{6}\left(17^{3/2}-5^{3/2}\right)$$

- You could have also distributed the $\sqrt{x}$

$$\sqrt{x}\sqrt{1+\frac{1}{4x}} = \sqrt{x\left(1+\frac{1}{4x}\right)} = \sqrt{x+\frac{1}{4}}$$

<u>Probabilities</u>

Continious random variable - A variable with a contitious range that usually represents measurements like height, weight, or time.

Probability density function - Describes how contitious random variables are distributed.

Continious random variable.

$$P(a \leq X \leq b) = \int_a^b f(x)\,dx$$

Always use intervals

Probability density function

Probability between 0 and 1

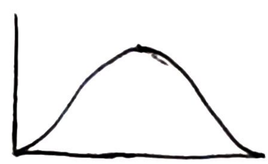
Mean/Average

$$\mu = \int_{-\infty}^{\infty} x f(x)\,dx$$

$$\int_{-\infty}^{\tilde{\mu}} f(x)\,dx = \frac{1}{2}$$ where $\tilde{\mu}$ is the median

The two most common probability density functions:

Normal distribution: $\frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}$



Exponential decay: $\frac{1}{\mu e^{x/\mu}}$

Ex: $f(x) = 0.006x(10-x)$ for $0 \leq x \leq 10$ and $f(x) = 0$ for all other values of x.

a) Verify f is a density function.

1. $f(x) \geq 0$ for all values of x
2. $\int_{-\infty}^{\infty} 0.006x(10-x)\,dx = 1?$ $\Rightarrow 0.006\int_0^{10} 10x - x^2\,dx$

$$= 0.006\left[5x^2 - \frac{x^3}{3}\right]_0^{10} = 0.006\left(500 - \frac{1000}{3}\right) = 0.006\left(\frac{500}{3}\right) = 1$$

b) Find $P(4 \leq X \leq 8) = \int_4^8 0.006x(10-x)\,dx =$