

Finding Integrals

1. Limit definition

$$\int_a^b f(x) dx = \lim_{n \rightarrow \infty} \overbrace{\sum_{i=1}^n f(x_i) \Delta x}^{\text{Riemann sum}}$$

$f(x_i) = \text{Height}$

$\Delta x = \text{Width} = \frac{b-a}{n}$

$x_i = a + i\Delta x$

- We use an infinite number of rectangles to get the area under the curve.

- If we don't use an infinite number, we can use the left, right, midpoint, smallest y value (lower sum), or largest y value (upper sum) to estimate the area under the curve.

- We can also get a rough estimate by finding the $\max(M)$ and $\min(m)$ of $f(x)$.

$$\text{If } m \leq f(x) \leq M, \text{ then } m(b-a) \leq \int_a^b f(x) dx \leq M(b-a)$$

$$\sum_{i=1}^n c = c \sum_{i=1}^n 1 = cn$$

$$\sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{i=1}^n a_i + b_i = \sum_{i=1}^n a_i + \sum_{i=1}^n b_i$$

$$\sum_{i=1}^n i = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^n i^3 = \left(\frac{n(n+1)}{2}\right)^2$$

$$\sum_{i=1}^n a_i - b_i = \sum_{i=1}^n a_i - \sum_{i=1}^n b_i$$

2. Properties of integrals

$$\int_a^b f(x) dx = - \int_b^a f(x) dx$$

$$\int_a^a f(x) dx = 0$$

$$\int_a^b c f(x) dx = c \int_a^b f(x) dx$$

$$\int_a^b f(x) dx + \int_b^c f(x) dx = \int_a^c f(x) dx$$

$$\int_a^b (f(x) + g(x)) dx = \int_a^b f(x) dx + \int_a^b g(x) dx$$

$$\int_a^b (f(x) - g(x)) dx = \int_a^b f(x) dx - \int_a^b g(x) dx$$

3. Fundamental theorem of calculus

Part 1: Deriving the integral of f gives F

$$\frac{d}{dx} \int_a^x f(t) dt = f(x) \quad a \text{ is any constant}$$

Ex: $\frac{d}{dx} \int_1^{x^4} \sec t \, dt$

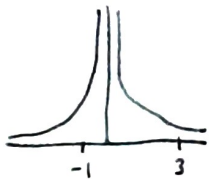
$$u = x^4 \quad \frac{d}{dx} = \frac{du}{dx} \cdot \frac{d}{du}$$
$$\frac{du}{dx} = 4x^3$$

$$4x^3 \cdot \frac{d}{du} \int_1^u \sec t \, dt = 4x^3 \cdot \sec u = 4x^3 \sec x^4$$

Part 2:

$$\int_a^b f(x) dx = F(x) \Big|_a^b = F(b) - F(a)$$

Ex: $\int_{-1}^3 \frac{1}{x^2} dx = -\frac{1}{x} \Big|_{-1}^3 = -\frac{4}{3}$



$f(x)$ has to be continuous or have a finite number of jump discontinuities over $[a, b]$ to be integrable.

Integration and differentiation are inverse processes.

4 U-Substitution

1. Define u

2. Find $\frac{du}{dx}$

3. Solve for dx

4. Plug u and dx into the integral and simplify

If definite integral: Re-write bounds for u

5. Solve integral

If indefinite integral: Substitute u so it's in terms of x .

$$\text{Ex 1: } \int x^3 \cos(x^4+2) dx \quad u = x^4 + 2$$

$$\frac{du}{dx} = 4x^3 \Rightarrow dx = \frac{du}{4x^3}$$

$$\int x^3 \cos(u) \frac{du}{4x^3} = \int \frac{1}{4} \cos u \, du = \frac{1}{4} \sin u + C = \frac{1}{4} \sin(x^4+2) + C$$

$$\text{Ex 2: } \int \sqrt{1+x^2} x^5 dx \quad u = 1+x^2 \quad u-1 = x^2$$

$$\frac{du}{dx} = 2x \Rightarrow dx = \frac{du}{2x}$$

$$\int \sqrt{u} x^5 \frac{du}{2x} = \int \sqrt{u} x^4 \frac{1}{2} du = \int \frac{1}{2} \sqrt{u} (u-1)^2 du$$

$$= \int \frac{1}{2} \sqrt{u} (u^2 - 2u + 1) du = \int \frac{1}{2} u^{\frac{5}{2}} - u^{\frac{3}{2}} + \frac{1}{2} u^{\frac{1}{2}} du$$

$$= \frac{1}{2} \cdot \frac{2}{\frac{7}{2}} u^{\frac{7}{2}} - \frac{2}{\frac{5}{2}} u^{\frac{5}{2}} + \frac{1}{2} \cdot \frac{2}{\frac{3}{2}} u^{\frac{3}{2}} + C = \frac{1}{7} (1+x^2)^{\frac{7}{2}} - \frac{2}{5} (1+x^2)^{\frac{5}{2}} + \frac{1}{3} (1+x^2)^{\frac{3}{2}} + C$$

$$\text{Ex 3: } \int \tan x \, dx = \int \frac{\sin x}{\cos x} dx \quad u = \cos x$$

$$\frac{du}{dx} = -\sin x \Rightarrow dx = -\frac{du}{\sin x}$$

$$\int \frac{\sin x}{u} \left(-\frac{du}{\sin x}\right) = \int -\frac{1}{u} du = -\ln|u| + C = -\ln|\cos x| + C$$

$$\text{Ex 4: } \int_1^e \frac{\ln x}{x} dx \quad u = \ln x$$

$$\frac{du}{dx} = \frac{1}{x} \Rightarrow dx = x \, du$$

$$\text{Lower: } u = \ln(1) = 0 \quad \text{Upper: } u = \ln(e) = 1$$

$$\int_0^1 \frac{u}{x} x \, du = \int_0^1 u \, du = \frac{1}{2} u^2 \Big|_0^1 = \frac{1}{2} (1^2) - \frac{1}{2} (0^2) = \frac{1}{2}$$

5. Symmetry: If $f(x)$ is continuous on $[-a, a]$

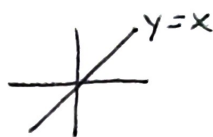
If $f(-x) = f(x)$ then



$$\int_{-a}^a f(x) dx = 2 \int_0^a f(x) dx$$

If $f(-x) = -f(x)$ then

$$\int_{-a}^a f(x) dx = 0$$



6. Integration by parts

Just like U-substitution is the inverse of the chain rule, integration by parts is the inverse of the product rule.

$$\frac{d}{dx} (f(x)g(x)) = f(x)g'(x) + f'(x)g(x)$$

$$f(x)g(x) = \int f(x)g'(x) dx + \int f'(x)g(x) dx$$

$$\int f(x)g'(x) dx = f(x)g(x) - \int f'(x)g(x) dx$$

$$\begin{aligned} u &= f(x) & dv &= g'(x) dx \\ du &= f'(x) dx & v &= g(x) \end{aligned}$$

$$\boxed{\int u dv = uv - \int v du}$$

Use LIPET for picking u :

L - $\ln$

I - Inverse trig

P - Powers of x

E - Exponential

T - Trig

The goal of integration by parts is to convert the integral to a simpler integral.

This may not always be the case.

$$\text{Definite integrals: } \int_a^b u dv = uv \Big|_a^b - \int_a^b v du$$

$$\text{Ex 1: } \int x \sin x \, dx \quad \begin{array}{ll} u = x & dv = \sin x \, dx \\ du = dx & v = -\cos x \end{array}$$

$$= x(-\cos x) - \int -\cos x \, dx = -x \cos x + \sin x + C$$

$$\text{Ex 2: } \int \ln x \, dx \quad \begin{array}{ll} u = \ln x & dv = dx \\ du = \frac{1}{x} dx & v = x \end{array}$$

$$= \ln x(x) - \int x \frac{1}{x} dx = x \ln x - x + C$$

$$\text{Ex 3: } \int t^2 e^t \, dt \quad \begin{array}{ll} u = t^2 & dv = e^t \, dt \\ du = 2t \, dt & v = e^t \end{array}$$

$$= t^2 e^t - \int e^t 2t \, dt = t^2 e^t - 2 \int t e^t \, dt \quad \begin{array}{ll} u = t & dv = e^t \, dt \\ du = dt & v = e^t \end{array}$$

$$= t^2 e^t - 2 \left[t e^t - \int e^t \, dt \right] = t^2 e^t - 2 t e^t + 2 e^t + C$$

$$\text{Ex 4: } \int e^x \sin x \, dx \quad \begin{array}{ll} u = e^x & dv = \sin x \, dx \\ du = e^x \, dx & v = -\cos x \end{array}$$

$$= e^x(-\cos x) - \int -\cos x e^x \, dx = -e^x \cos x + \int \cos x e^x \, dx$$

$$\begin{array}{ll} u = e^x & dv = \cos x \, dx \\ du = e^x \, dx & v = \sin x \end{array}$$

$$\int e^x \sin x \, dx = -e^x \cos x + e^x \sin x - \int \sin x e^x \, dx \quad \leftarrow \text{Add to the other side.}$$

$$2 \int e^x \sin x \, dx = -e^x \cos x + e^x \sin x$$

$$\int e^x \sin x \, dx = \frac{1}{2} e^x (\sin x - \cos x) + C$$

7. Powers of sin and cos

$$\int \sin^m x \cos^n x dx$$

$\sin^m x$	$\cos^n x$	Solution
Odd	Odd	Either
Odd	Even	$u = \cos x$
Even	Odd	$u = \sin x$
Even	Even	$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$ $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$

$$\left. \begin{array}{l} \text{Odd} \\ \text{Odd} \\ \text{Even} \end{array} \right\} \sin^2 x + \cos^2 x = 1$$

Ex 1: $\int \cos^5 x dx$ $\cos^5 x$ is odd, $\sin^0 x$ is even

$$u = \sin x$$

$$du = \cos x dx \Rightarrow dx = \frac{du}{\cos x}$$

$$\int \cos^5 x \frac{du}{\cos x} = \int \cos^4 x du = \int (\cos^2 x)^2 du$$

$$\cos^2 x = 1 - \sin^2 x \quad = \int (1 - \sin^2 x)^2 du = \int (1 - u^2)^2 du$$

$$(1 - u^2)(1 - u^2) = 1 - 2u^2 + u^4 \quad = \int 1 - 2u^2 + u^4 du$$

$$= u - \frac{2}{3}u^3 + \frac{1}{5}u^5 + C = \sin x - \frac{2}{3}\sin^3 x + \frac{1}{5}\sin^5 x + C$$

Ex 2: $\int \sin^4 x dx$ $\sin^4 x$ is even and $\cos^0 x$ is even

$$\sin^4 x = (\sin^2 x)^2 = \left(\frac{1}{2}(1 - \cos 2x)\right)^2 = \frac{1}{4}(1 - \cos 2x)^2$$

$$= \int \frac{1}{4}(1 - \cos 2x)^2 dx = \frac{1}{4} \int 1 - 2\cos 2x + \cos^2 2x dx$$

$$= \frac{1}{4} \left[\int dx - 2 \int \cos 2x dx + \int \cos^2 2x dx \right]$$

$$= \frac{1}{4} \left[x - \sin 2x + \int \cos^2 2x dx \right]$$

$$\cos^2(2x) = \frac{1}{2} (1 + \cos 4x)$$

$$= \frac{1}{4} \left[x - \sin 2x + \frac{1}{2} \int 1 + \cos 4x dx \right]$$

$$= \frac{1}{4} x - \frac{1}{4} \sin 2x + \frac{1}{8} \left[\int dx + \int \cos 4x dx \right]$$

$$= \frac{1}{4} x - \frac{1}{4} \sin 2x + \frac{1}{8} x + \frac{1}{32} \sin 4x + C = \frac{3}{8} x - \frac{1}{4} \sin 2x + \frac{1}{32} \sin 4x + C$$

8. Powers of tan and sec

$$\int \tan^m x \sec^n x dx$$

This also applies for cot and csc

$\tan^m x$	$\sec^n x$	Solution
Odd	Odd	$u = \sec x$
Odd	Even	Either
Even	Odd	other
Even	Even	$u = \tan x$

$$\tan^2 x + 1 = \sec^2 x$$

- Integration by parts
- $\int \sec x = \ln |\sec x + \tan x| + C$
- $\int \tan x = \ln |\sec x| + C$
- Move to other side and divide

Ex 1: $\int \tan^2 x \sec^4 x dx$

$$u = \tan x$$

$$du = \sec^2 x dx \Rightarrow dx = \frac{du}{\sec^2 x}$$

$$\int \tan^2 x \sec^4 x \frac{du}{\sec^2 x} = \int u^2 \sec^2 x du$$

$$\sec^2 x = 1 + \tan^2 x = \int u^2 (1 + u^2) du = \int u^2 + u^4 du$$

$$= \frac{1}{3} u^3 + \frac{1}{5} u^5 + C = \frac{1}{3} \tan^3 x + \frac{1}{5} \tan^5 x + C$$

Ex 2: $\int \sec^3 x dx = \int \sec^2 x \sec x dx = \int (\tan^2 x + 1) \sec x dx$

$$= \int \tan^2 x \sec x + \sec x \, dx = \int \tan^2 x \sec x \, dx + \int \sec x \, dx$$

$$\int \sec x \, dx = \int \frac{\sec x}{1} \cdot \frac{\sec x + \tan x}{\sec x + \tan x} \, dx = \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} \, dx$$

$$u = \sec x + \tan x \quad du = \sec x \tan x + \sec^2 x \, dx$$

$$= \int \frac{\sec^2 x + \sec x \tan x}{u} \cdot \frac{du}{\sec^2 x + \sec x \tan x} = \int \frac{1}{u} du = \ln |\sec x + \tan x| + C$$

$$\int \tan^2 x \sec x \, dx \quad \begin{array}{ll} u = \tan x & du = \sec x \tan x \, dx \\ du = \sec^2 x \, dx & v = \sec x \end{array}$$

$$= \sec x \tan x - \int \sec^3 x \, dx$$

$$\int \sec^3 x \, dx = \sec x \tan x - \int \sec^3 x \, dx + \ln |\sec x + \tan x| + C$$

$$= \frac{1}{2} \sec x \tan x + \frac{1}{2} \ln |\sec x + \tan x| + C$$

9. $\sin A$ and $\cos B$

$$\int \sin A \cos B \, dx \quad \text{or} \quad \int \sin A \sin B \, dx \quad \text{or} \quad \int \cos A \cos B \, dx$$

$$\sin A \cos B = \frac{1}{2} [\sin(A-B) + \sin(A+B)]$$

$$\sin A \sin B = \frac{1}{2} [\cos(A-B) - \cos(A+B)]$$

$$\cos A \cos B = \frac{1}{2} [\cos(A-B) + \cos(A+B)]$$

$$\int \sin 4x \cos 5x \, dx = \int \frac{1}{2} [\sin(-x) + \sin(9x)] \, dx$$

$$= \frac{1}{2} \left[\int -\sin x \, dx + \int \sin(9x) \, dx \right] = \frac{1}{2} \left[\cos x - \frac{1}{9} \cos 9x \right] + C$$

10. Trig Inverse Substitution

Expression in integrand

$$\sqrt{a^2 - x^2}$$

$$\sqrt{a^2 + x^2}$$

$$\sqrt{x^2 - a^2}$$

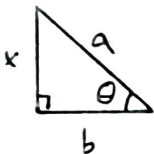
Trig Substitution

$$x = a \sin \theta$$

$$x = a \tan \theta$$

$$x = a \sec \theta$$

$$x = a \sin \theta \Rightarrow \sin \theta = \frac{x}{a} = \frac{\text{OPP}}{\text{hyp}}$$



$$\text{Pythagorean: } a^2 = x^2 + b^2 \Rightarrow b^2 = a^2 - x^2 \Rightarrow b = \sqrt{a^2 - x^2}$$

$$\text{Ex 1: } \int \frac{\sqrt{9-x^2}}{x^2} dx$$

$$x = 3 \sin \theta$$

$$dx = 3 \cos \theta d\theta$$

$$\sqrt{9-x^2} = \sqrt{3^2-x^2} = \sqrt{3^2-(3 \sin \theta)^2} = \sqrt{3^2-3^2 \sin^2 \theta} = \sqrt{3^2(1-\sin^2 \theta)}$$

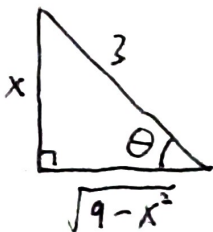
$$= \sqrt{3^2 \cos^2 \theta} = 3 |\cos \theta|$$

$$\int \frac{\sqrt{9-x^2}}{x^2} dx = \int \frac{3 |\cos \theta|}{3^2 \sin^2 \theta} 3 \cos \theta d\theta = \int \frac{\cos^2 \theta}{\sin^2 \theta} d\theta$$

$$= \int \cot^2 \theta d\theta = \int \csc^2 \theta - 1 d\theta = -\cot \theta - \theta + C$$

$$x = 3 \sin \theta \Rightarrow \frac{x}{3} = \sin \theta \Rightarrow \theta = \sin^{-1}\left(\frac{x}{3}\right)$$

soh cah tan



$$\cot \theta = \frac{\text{adj}}{\text{opp}} = \frac{\sqrt{9-x^2}}{x}$$

$$= -\frac{\sqrt{9-x^2}}{x} - \sin^{-1}\left(\frac{x}{3}\right) + C$$

$$\text{Ex 2: } \int \frac{x}{\sqrt{3-2x-x^2}} dx$$

$$3-2x-x^2 = 3-(2x+x^2) = 3+1-(2x+x^2+1) = 4-(x+1)^2$$

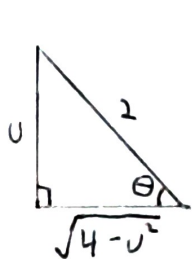
$$= \int \frac{x}{\sqrt{2^2-(x+1)^2}} dx \quad \begin{array}{l} u=x+1 \Rightarrow x=u-1 \\ du=dx \end{array}$$

$$= \int \frac{u-1}{\sqrt{2^2-u^2}} du \quad \begin{array}{l} u=2\sin\theta \\ du=2\cos\theta d\theta \end{array}$$

$$\sqrt{2^2-2^2\sin^2\theta} = \sqrt{2^2(1-\sin^2\theta)} = \sqrt{2^2\cos^2\theta} = 2|\cos\theta|$$

$$= \int \frac{2\sin\theta-1}{2|\cos\theta|} 2\cos\theta d\theta = \int 2\sin\theta-1 d\theta = -2\cos\theta - \theta + C$$

$$u=2\sin\theta \Rightarrow \sin\theta = \frac{u}{2} \Rightarrow \theta = \sin^{-1}\left(\frac{u}{2}\right)$$



$$\cos\theta = \frac{\sqrt{4-u^2}}{2}$$

$$= -2 \frac{\sqrt{4-u^2}}{2} - \sin^{-1}\left(\frac{u}{2}\right) + C$$

$$= -\sqrt{4-(x+1)^2} - \sin^{-1}\left(\frac{x+1}{2}\right) + C$$

$$= -\sqrt{3-2x-x^2} - \sin^{-1}\left(\frac{x+1}{2}\right) + C$$

11. Method of partial fractions

This method works for rational functions which are $\frac{\text{polynomial}}{\text{polynomial}}$

①. Sometimes you need to do a U-sub on $\sqrt{g(x)}$

$$\text{Ex: } \int \frac{\sqrt{x+4}}{x} dx \quad u = \sqrt{x+4} \quad x = u^2 - 4 \quad = 2 \int \frac{u^2}{u^2 - 4} du$$

$$du = \frac{1}{2\sqrt{x+4}} dx \quad dx = 2u du$$

1. If the degree of the numerator is $\geq$ degree of denominator, then use long division.

$$\text{Ex: } \int \frac{5x^2}{x^2 + x + 1} dx \quad \frac{\text{numerator}}{\text{denominator}} = \int \frac{\text{answer}}{\text{denominator}} + \frac{\text{remainder}}{\text{denominator}}$$

$$\begin{array}{r} 5 \\ x^2 + x + 1 \overline{) 5x^2 + 0x + 0} \\ \underline{-(5x^2 + 5x + 5)} \\ -5x - 5 \end{array} \quad = \int 5 + \frac{-5x-5}{x^2+x+1} dx$$

2. Factor the denominator as far as possible.

$$\text{Ex: } \int \frac{1}{(x^2-1)^2} dx = \int \frac{1}{((x-1)(x+1))^2} dx = \int \frac{1}{(x-1)^2(x+1)^2} dx$$

- Sometimes you can try numbers and see if they equal 0.

$$\text{Ex: } \int \frac{1}{x^3 + x^2 + x + 1} dx \quad \text{Try } x = -1: -1 + 1 - 1 + 1 = 0$$

therefore $(x+1)$ is a factor.

$$\frac{x^2+1}{x+1 \overline{) x^3+x^2+x+1}} = \int \frac{1}{(x+1)(x^2+1)} dx$$

- You also may need to find the square.

$$\text{Ex: } \int \frac{1}{4x^2 - 4x + 3} dx = \int \frac{1}{(2x-1)^2 + 2} dx$$

3. Find the partial fraction decomposition.

$$\frac{\text{numerator}}{(ax+b)^n} = \frac{A}{ax+b} + \frac{B}{(ax+b)^2} + \dots + \frac{C}{(ax+b)^n}$$

$$\frac{\text{numerator}}{(ax^2+bx+c)^n} = \frac{Ax+B}{ax^2+bx+c} + \frac{Cx+D}{(ax^2+bx+c)^2} + \dots + \frac{Ex+F}{(ax^2+bx+c)^n}$$

4. Add all coefficients with a least common denominator.

5. Set numerators equal to each other.

6. Find coefficients (A, B, C, ...)

A) Plug in a value for x, preferably one that sets other coefficients to 0.

Ex: $1 = A(x+1) + B(x-1)$

Try $x=1$: $1 = A(2) + B(0) \Rightarrow A = \frac{1}{2}$

Try $x=-1$: $1 = A(0) + B(-2) \Rightarrow B = -\frac{1}{2}$

B) Expand, add like terms together, set polynomial terms equal.

$$1 = Ax + A + Bx - B$$

$$0x + 1 = (A+B)x + (A-B)$$

$$0 = A+B \quad \text{and} \quad 1 = A-B$$

$$A = 1+B \Rightarrow 0 = 1+B+B = 1+2B$$

$$-1 = 2B \quad B = -\frac{1}{2} \quad \text{and} \quad A = \frac{1}{2}$$

7. Write coefficients in the integral and solve.

Ex: $\int \frac{x^4 - 2x^2 + 4x + 1}{x^3 - x^2 - x + 1} dx$

$$\begin{array}{r} x+1 \\ x^3 - x^2 - x + 1 \overline{) x^4 + 0x^3 - 2x^2 + 4x + 1} \\ \underline{-(x^4 - x^3 - x^2 + x)} \\ x^3 - x^2 + 3x + 1 \\ \underline{-(x^3 - x^2 - x + 1)} \\ 4x \end{array}$$

$$= \int x+1 + \frac{4x}{x^3 - x^2 - x + 1} dx$$

$$= \frac{1}{2}x^2 + x + \int \frac{4x}{x^3 - x^2 - x + 1} dx$$

$$(1) \int \frac{4x}{x^3 - x^2 - x + 1} dx$$

$$= \int \frac{4x}{(x+1)(x-1)^2} dx$$

Try $x = -1$: $-1 - 1 + 1 + 1 = 0$

so $(x+1)$ is a factor

$$\begin{array}{r} x^2 - 2x + 1 \\ x+1 \overline{) x^3 - x^2 - x + 1} \\ \underline{-(x^3 + x^2)} \\ -2x^2 - x \\ \underline{-(-2x^2 - 2x)} \\ x + 1 \end{array}$$

Try $x = 1$: $1 - 2 + 1 = 0$

so $(x-1)$ is a factor

$$x^2 - 2x + 1 = (x-1)(x-1)$$

$$= \int \frac{A}{x+1} + \frac{B}{x-1} + \frac{C}{(x-1)^2} dx$$

$$\frac{4x}{(x+1)(x-1)^2} = \frac{A(x-1)^2 + B(x-1)(x+1) + C(x+1)}{(x+1)(x-1)^2}$$

$$4x = A(x-1)^2 + B(x-1)(x+1) + C(x+1)$$

$x = 1$: $4 = A(0) + B(0)(2) + C(2) = 2C \Rightarrow C = 2$

$x = -1$: $-4 = A(-2)^2 + B(-2)(0) + C(0) = 4A \Rightarrow A = -1$

$x = 0$: $0 = A(-1)^2 + B(-1)(1) + C(1) = A - B + C = -1 - B + 2$

$0 = 1 - B \Rightarrow B = 1$

$$= \int \frac{-1}{x+1} + \frac{1}{x-1} + \frac{2}{(x-1)^2} dx$$

$$= -\int \frac{1}{x+1} dx + \int \frac{1}{x-1} dx + 2 \int \frac{1}{(x-1)^2} dx$$

$$= -\ln|x+1| + \ln|x-1| - 2 \cdot \frac{1}{x-1} + C$$

① $= \frac{1}{2}x^2 + x - \ln|x+1| + \ln|x-1| - \frac{2}{x-1} + C$

12. Trapezoidal and Midpoint Rules

Used to approximate definite integrals and gives a better approx with fewer points than left or right endpoints.

$$\text{Trapezoidal: } \frac{\text{left} + \text{right}}{2} = \frac{\Delta x}{2} (f(x_0) + 2f(x_1) + 2f(x_2) + \dots + f(x_n))$$

$$\text{Midpoint: } \Delta x (f(a + \frac{\Delta x}{2}) + f(a + \frac{\Delta x}{2} + \Delta x) + \dots + f(a + \frac{\Delta x}{2} + n\Delta x))$$

$$\text{Ex: } \int_0^1 e^{x^2} dx \quad n=4 \quad \Delta x = \frac{b-a}{n} = \frac{1-0}{4} = \frac{1}{4} = \frac{2}{8} \quad \frac{\Delta x}{2} = \frac{1}{8}$$

$$\text{Trap: } \frac{\Delta x}{2} (f(0) + 2f(\frac{1}{4}) + 2f(\frac{2}{4}) + 2f(\frac{3}{4}) + f(1)) = 1.49068$$

$$\text{Mid: } \Delta x (f(\frac{1}{8}) + f(\frac{3}{8}) + f(\frac{5}{8}) + f(\frac{7}{8})) = 1.44875$$

Actual answer: 1.46265 * The Midpoint tends to be more accurate.

13. Improper integrals

Either bound is $\pm \infty$ and $f(x)$ is continuous on bounds.

$$\frac{\int_a^\infty f(x) dx = \lim_{b \rightarrow \infty} \int_a^b f(x) dx}{\int_{-\infty}^b f(x) dx = \lim_{a \rightarrow -\infty} \int_a^b f(x) dx}$$

$$\frac{\int_{-\infty}^\infty f(x) dx = \lim_{a \rightarrow -\infty} \int_a^c f(x) dx + \lim_{b \rightarrow \infty} \int_c^b f(x) dx}$$

c is usually chosen to be 0 or 1.

$$\text{Ex: } \int_{-\infty}^\infty \frac{1}{1+x^2} dx = \lim_{a \rightarrow -\infty} \int_a^0 f(x) dx + \lim_{b \rightarrow \infty} \int_0^b f(x) dx$$

$$= \lim_{a \rightarrow -\infty} [\tan^{-1} x]_a^0 + \lim_{b \rightarrow \infty} [\tan^{-1} x]_0^b = 0 - (-\frac{\pi}{2}) + \frac{\pi}{2} - 0 = \pi$$

Vertical asymptotes

On left endpoint: $\int_a^b f(x) dx = \lim_{c \rightarrow a^+} \int_c^b f(x) dx$

On right endpoint: $\int_a^b f(x) dx = \lim_{c \rightarrow b^-} \int_a^c f(x) dx$

In the center: $\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx$

Ex: $\int_1^4 \frac{1}{x-2} dx$ $x \neq 2$ and it's in $[1, 4]$

$$= \int_1^2 \frac{1}{x-2} dx + \int_2^4 \frac{1}{x-2} dx = \lim_{c \rightarrow 2^-} \int_1^c \frac{1}{x-2} dx + \lim_{c \rightarrow 2^+} \int_c^4 \frac{1}{x-2} dx$$

$$= \lim_{c \rightarrow 2^-} \left[\ln|x-2| \right]_1^c + \lim_{c \rightarrow 2^+} \left[\ln|x-2| \right]_c^4$$

$$= \lim_{c \rightarrow 2^-} (\ln|c-2| - \ln 1) + \lim_{c \rightarrow 2^+} (\ln 2 - \ln|c-2|)$$

$$= -\infty + \ln 2 + \infty = \text{DNE. If one side goes to infinity, then it DNE.}$$

Comparison test.

If $f(x)$ is on top of $g(x)$ and they're both continuous on $[a, \infty)$

$\int_a^\infty f(x) dx$ converges	$= \int_a^\infty g(x) dx$ converges	Isolate the dominate terms: Ex: $\int_0^\infty \frac{1}{x+1} dx \rightarrow \frac{1}{x}$ Ex: $\int_0^1 \frac{\sec^2 x}{\sqrt{x}(x+1)} dx \rightarrow \frac{1}{x^{3/2}}$
$\int_a^\infty g(x) dx$ diverges	$= \int_a^\infty f(x) dx$ diverges	

compare

If $f(x)$ and $g(x)$ grow at similar rates and either one converges/diverges, then the other converges/diverges.

$$\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = L, \quad 0 < L < \infty, \quad \text{then } \int_a^\infty f(x) dx \text{ and } \int_a^\infty g(x) dx$$

both converge or diverge.

$$\text{Ex: } \int_1^\infty \frac{1}{e^x - 5} dx \quad \lim_{x \rightarrow \infty} \frac{\frac{1}{e^x}}{\frac{1}{e^x - 5}} = \lim_{x \rightarrow \infty} \frac{e^x - 5}{e^x} = \lim_{x \rightarrow \infty} 1 - \frac{5}{e^x} = 1 > 0$$

$\int_1^\infty \frac{1}{e^x} dx$ converges so $\int_1^\infty \frac{1}{e^x - 5} dx$ also converges.